FACTORIZATION THROUGH MATRIX SPACES FOR FINITE RANK OPERATORS BETWEEN C^*-ALGEBRAS

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0. Introduction. In this paper we consider factorizations of finite rank operators through finite-dimensional C^*-algebras. We are interested in factorization norms involving either the completely bounded norm $\| \ \|_{\mathrm{cb}}$ or Haagerup's decomposable norm $\| \ \|_{\mathrm{dec}}$ (see [11]). Let us denote by M_n the C^*-algebra of all $n \times n$ matrices with complex entries. Let A and B be two C^*-algebras, and let us consider a finite rank bounded operator $u\colon A \to B$. Then for n large enough, say $n \geq rk(u)$, we may write factorizations of the form $u = \beta\alpha$, for some bounded operators

$$A \xrightarrow{\alpha} M_n \xrightarrow{\beta} B. \tag{0.1}$$

Our main result (Theorem 2.1) says that for any $\varepsilon > 0$, one can find α and β as above such that $\|\alpha\|_{\mathrm{cb}}\|\beta\|_{\mathrm{dec}} \leq (1+\varepsilon)\|u\|_{\mathrm{dec}}$. If u is completely positive, then $\|u\|_{\mathrm{dec}} = \|u\|$; hence in that case, we obtain that $\|u\| = \inf\{\|\alpha\|_{\mathrm{cb}}\|\beta\|_{\mathrm{dec}}\}$, where the infimum runs over all factorizations as above. This new result that finite rank, completely positive maps factor through matrix algebras gives some explanation of the phenomenon behind the classical result of Choi-Effros-Kirchberg [4], [16] characterizing nuclear C^*-algebras either by the completely positive approximation property or equivalently by the *approximate matriciality* of the algebra.

For a finite rank operator $u\colon A \to B$ between C^*-algebras, let us now introduce $\gamma(u) = \inf\{\|\alpha\|_{\mathrm{cb}}\|\beta\|_{\mathrm{cb}}\}$, where the infimum runs over all $n \geq 1$ and all factorizations of u of the form (0.1). We obviously have $\|u\|_{\mathrm{cb}} \leq \gamma(u)$. In Section 3 we consider the natural problem of whether the converse inequality holds, that is, $\|u\|_{\mathrm{cb}} = \gamma(u)$. We show that this holds if B has the weak expectation property (as defined in [18]). In the case when B is a von Neumann algebra, we obtain the following characterization. The equality $\|u\|_{\mathrm{cb}} = \gamma(u)$ holds for all A and all u if and only if B is injective. This result shows in particular that the two norms $\gamma(\ )$ and $\| \ \|_{\mathrm{cb}}$ may be different. Thus the decomposable norm $\| \ \|_{\mathrm{dec}}$ behaves better than the completely bounded one when dealing with factorization through matrix algebras.

Our factorization results have several applications to the theory of operator spaces. First, we give a new proof of the recent theorem by Effros-Junge-Ruan [6], which asserts that given any von Neumann algebra R, the predual operator space R_* is locally reflexive in the sense of [5], [7]. Second, we show that given any two C^*-algebras A and B, the Banach spaces of completely integral maps and completely 1-summing

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maps (in the sense of [9], [10], [30]) from B into A coincide provided that B has the weak expectation property. Third, in Section 4, we study factorization through M_n for finite rank operators defined on spaces of vector-valued Hankel operators. We extend a result due to the first author that says that any finite rank operator defined on the operator space Ha of (scalar-valued) Hankel operators is exact (see [14]).

1. Preliminary background. We recall here some standard notation and terminology. We refer the reader to [24] for general information on completely bounded maps and to [1], [2], [7], [8], [10], [29], [30], [31] for the theory of operator spaces. By definition, an operator space is a closed subspace $X \subset B(H)$ of the C^*-algebra of all bounded operators on a complex Hilbert space H. We denote by $M_n(X)$ the space of all $n \times n$ matrices with entries in X, which is equipped with the norm inherited by the inclusion $M_n(X) \subset B(\ell_2^n(H))$. In that definition, $\ell_2^n(H)$ is the Hilbertian direct sum of n copies of H. Let $X \subset B(H)$ and $Y \subset B(K)$ be two operator spaces. We denote by $CB(X,Y)$ the Banach space of all completely bounded maps from X into Y, equipped with the completely bounded norm $\| \ \|_{\rm cb}$. We say that a bounded operator $u\colon X \to Y$ is completely contractive (or is a complete contraction) when $\|u\|_{\rm cb} \leq 1$, and that it is completely isometric (or is a complete isometry) when $I_{M_n} \otimes u$ is an isometry from $M_n(X)$ into $M_n(Y)$ for all $n \geq 1$. The minimal (or spatial) tensor product is denoted by $X \otimes_{\min} Y$. We recall that it may be defined as the closure of $X \otimes Y$ into the C^*-algebra $B(H \overset{2}{\otimes} K)$, where $H \overset{2}{\otimes} K$ is the Hilbertian tensor product of H and K. The projective operator space tensor product (for which we refer to [2], [7], [8]) is denoted by $X \widehat{\otimes} Y$. When $X = A$ and $Y = B$ are C^*-algebras, a third tensor product plays a crucial role, namely, the maximal tensor product $A \otimes_{\max} B$ defined as the completion of $A \otimes B$ under its greatest C^*-norm (see, e.g., [32] or [24]).

Let X be an operator space. We recall that its dual space X^* can be regarded as an operator space, once equipped with the so-called standard dual operator space structure (see [1], [2], [7], [8]). The latter is characterized by the fact that for any other operator space Y, the canonical identification of $X^* \otimes Y$ with the space of finite rank operators from X into Y yields an isometric embedding

$$X^* \otimes_{\min} Y \subset CB(X,Y). \tag{1.1}$$

As another crucial identity, we recall that for any X, Y, we have

$$CB(X,Y^*) = \big(X \widehat{\otimes} Y\big)^*. \tag{1.2}$$

We use the notion of quotient operator space structure, which was introduced in [31]. Let X and Y be two operator spaces such that $Y \subset X$ completely isometrically. Then the quotient operator space structure on X/Y is defined by letting $M_n(X/Y) = M_n(X)/M_n(Y)$ for any integer $n \geq 1$. Note that with that definition, we have

$$\left(\frac{X}{Y}\right)^* = Y^\perp \subset X^* \quad \text{completely isometrically.} \tag{1.3}$$

Now let $R \subset B(H)$ be a von Neumann algebra. We can regard its predual R_* as an operator space by means of the canonical embedding $R_* \subset R^*$. Then it turns out that R is the standard dual operator space of R_* (see [1, Theorem 2.9]). In particular, the space $S_1(H) = B(H)_*$ of trace class operators on H can be regarded as an operator space. The operator space $S_1(\ell_2)$ is simply denoted by S_1 whereas we denote by $\mathcal{K}$ the C^*-algebra of all compact operators on ℓ_2. For any $n \geq 1$, we denote by S_1^n the dual operator space of M_n. Similarly, we denote by ℓ_∞^n and ℓ_1^n the commutative n-dimensional C^*-algebra and its dual operator space, respectively.

We end this section with a brief review of Haagerup's decomposable norm from [11]. Given any two C^*-algebras A and B, we say that a linear map $u\colon A \to B$ is decomposable if it lies in the linear span of completely positive maps. We denote by $D(A, B)$ the vector space of all decomposable maps from A into B. For any $u \in D(A, B)$, we let $\|u\|_{\text{dec}} = \inf\{\max\{\|S_1\|\|S_2\|\}\}$, where the infimum runs over all completely positive maps $S_1\colon A \to B$ and $S_2\colon A \to B$ such that the operator $v\colon A \to M_2(B)$ defined by

$$v(x) = \begin{pmatrix} S_1(x) & u(x^*)^* \\ u(x) & S_2(x) \end{pmatrix}$$

is completely positive. It is shown in [11] that $\|\ \|_{\text{dec}}$ is well defined and is a complete norm on $D(A, B)$. Moreover the inequality $\|u\|_{\text{cb}} \leq \|u\|_{\text{dec}}$ holds for any $u \in D(A, B)$. This inequality is actually an equality in two important cases. First,

$$\text{(1.4)} \qquad \text{if } u\colon A \to B \ \text{ is completely positive, then } \|u\|_{\text{dec}} = \|u\|_{\text{cb}} (= \|u\|).$$

Second, it follows from [33] and [23] that

$$\text{(1.5)} \qquad \text{if } B \text{ is injective, then } \|u\|_{\text{dec}} = \|u\|_{\text{cb}} \text{ for any } u \in D(A, B).$$

It is also proved in [11] that if C is a third C^*-algebra, then

(1.6)

$$\text{if } u \in D(A, C), v \in D(C, B), \text{ then } vu \in D(A, B) \text{ and } \|vu\|_{\text{dec}} \leq \|v\|_{\text{dec}} \|u\|_{\text{dec}}.$$

We conclude with a simple tensorization result that first appeared in [15].

LEMMA 1.1 [15]. *Let A, B, C be three C^*-algebras and let $u \in D(A, B)$. Then the mapping $u \otimes I_C\colon A \otimes C \longrightarrow B \otimes C$ extends (in a unique way) to a decomposable map $\widetilde{u}\colon A \otimes_{\max} C \longrightarrow B \otimes_{\max} C$, which satisfies $\|\widetilde{u}\|_{\text{dec}} \leq \|u\|_{\text{dec}}$.*

2. Factorization through M_n with respect to the decomposable norm. The following theorem is the key result of our paper.

THEOREM 2.1. *Let A and B be two C^*-algebras and let $u\colon A \to B$ be a finite rank bounded operator.*

(i) *For any* $\varepsilon > 0$, *there exist an integer* $n \geq 1$ *and two linear maps* $\alpha\colon A \to M_n$, $\beta\colon M_n \to B$ *such that* $u = \beta\alpha$ *and* $\|\alpha\|_{\mathrm{cb}}\|\beta\|_{\mathrm{dec}} \leq (1+\varepsilon)\|u\|_{\mathrm{dec}}$.

(ii) *Assume that* A *is a von Neumann algebra and that* u *is* w^**-continuous. Then (i) can be achieved for some* w^**-continuous* α.

In the course of the proof of Theorem 2.1, we use a representation theorem due to Effros and Ruan [7] that we now recall. Let R and M be two von Neumann algebras, and let $R\overline{\otimes}M$ be their von Neumann tensor product. Then the isometry (1.1) (with $X = R_*, Y = M$) extends to the isometric identification

$$CB(R_*, M) = R\overline{\otimes}M. \tag{2.1}$$

More precisely, given any φ in R_*, let $\widehat{\varphi}$ be the unique extension of $\varphi \otimes I_M$ as a w^*-continuous linear map from $R\overline{\otimes}M$ into M. Then (2.1) means that for any θ in $R\overline{\otimes}M$, the mapping $\sigma\colon R_* \to M$ defined by

$$\sigma(\varphi) = \widehat{\varphi}(\theta) \tag{2.2}$$

is completely bounded with $\|\sigma\|_{\mathrm{cb}} = \|\theta\|$, and that the resulting isometry $\theta \mapsto \sigma$ from $R\overline{\otimes}M$ into $CB(R_*, M)$ is surjective.

The proof of Theorem 2.1 essentially relies on a factorization result recently discovered by Pisier in [29]. Before stating it, we introduce some relevant notation. Let E_1, E_2 be two operator spaces and let $\mathcal{H}$ be a Hilbert space. Given two linear maps $\sigma_1\colon E_1 \to B(\mathcal{H})$, $\sigma_2\colon E_2 \to B(\mathcal{H})$, we define $\sigma_1{\cdot}\sigma_2\colon E_1 \otimes E_2 \to B(\mathcal{H})$ by letting

$$\sigma_1{\cdot}\sigma_2\Big(\sum x_i^1 \otimes x_i^2\Big) = \sum \sigma_1\big(x_i^1\big)\sigma_2\big(x_i^2\big),$$

for any finite families $(x_i^1)_i$ in E_1, $(x_i^2)_i$ in E_2.

Definition 2.2 [29, Section 6.3]. Let E be an operator space and let B be a C^*-algebra. For any $y \in E \otimes B$, we set

$$\delta(y) = \sup\big\{\|\sigma\cdot\pi(y)\|\big\},$$

where the supremum runs over all Hilbert spaces $\mathcal{H}$, all completely contractive maps $\sigma\colon E \to B(\mathcal{H})$, and all $*$-representations $\pi\colon B \to B(\mathcal{H})$ with commuting ranges (i.e., $\sigma(e)\pi(b) = \pi(b)\sigma(e)$ for any $e \in E, b \in B$). The completion of $E \otimes B$ under δ (which is obviously a norm) is denoted by $E \overset{\delta}{\otimes} B$.

Pisier's theorem given below expresses δ as a suitable factorization norm. The following statement is a combination of [29, Theorem 6.3.1] and [29, Corollary 6.3.5].

THEOREM 2.3 [29]. *Let* E *be an operator space, let* B *be a* C^**-algebra, and let* $y \in E \otimes B$. *Then*

$$\text{(i)} \qquad \delta(y) = \inf\left\{\big\|[e_{ij}]\big\|_{M_n(E)}\left\|\sum_{i=1}^n a_i a_i^*\right\|^{1/2}\left\|\sum_{j=1}^n b_j^* b_j\right\|^{1/2}\right\},$$

where the infimum runs over all decompositions of y of the form $y = \sum_{1\leq i,j\leq n} e_{ij} \otimes a_i b_j$, with arbitrary $n \geq 1$, $e_{ij} \in E$, and $a_i, b_j \in B$.

(ii) *Let $u\colon E^* \to B$ be the linear map represented by y. Then*

$$\delta(y) = \inf\left\{\|\alpha\|_{\mathrm{cb}}\|\beta\|_{\mathrm{dec}}\right\},$$

where the infimum runs over all factorizations $u = \beta\alpha$ where $\alpha\colon E^* \to M_n$ is w^*-continuous and $\beta\colon M_n \to B$.

(iii) *Assume that E is a dual operator space, that is, $E = F^*$ for some operator space F, and let $u\colon F \to B$ be the linear map represented by y. Then*

$$\delta(y) = \inf\left\{\|\alpha\|_{\mathrm{cb}}\|\beta\|_{\mathrm{dec}}\right\},$$

where the infimum runs over all factorizations $u = \beta\alpha$ with $\alpha\colon F \to M_n$ and $\beta\colon M_n \to B$.

Remark 2.4. Let $y \in E \otimes B$ as above, and let $j\colon B \to B^{**}$ be the canonical inclusion map. Then we have

$$\delta(y) = \delta\big(\big(I_E \otimes j\big)(y)\big). \tag{2.3}$$

Indeed, given a Hilbert space $\mathcal{H}$, let $\sigma\colon E \to B(\mathcal{H})$ and $\pi\colon B \to B(\mathcal{H})$ be a complete contraction and a $*$-representation, respectively, and assume that they have commuting ranges. Let $\pi_1\colon B^{**} \to B(\mathcal{H})$ be the normal extension of π; then $\pi_1(B^{**})$ commutes with $\sigma(E)$, and hence

$$\left\|\sigma \cdot \pi(y)\right\| = \left\|\sigma \cdot \pi_1\big(\big(I_E \otimes j\big)(y)\big)\right\| \leq \delta\big(\big(I_E \otimes j\big)(y)\big),$$

from which we get the inequality $\leq$ in (2.3). The converse inequality is obvious.

Remark 2.5. Let A, B be two C^*-algebras, and let $\alpha\colon A \to M_n$ and $\beta\colon M_n \to B$ be two bounded operators. Then it follows from (1.6) and (1.5) that

$$\|\beta\alpha\|_{\mathrm{dec}} \leq \|\alpha\|_{\mathrm{dec}}\|\beta\|_{\mathrm{dec}} \leq \|\alpha\|_{\mathrm{cb}}\|\beta\|_{\mathrm{dec}}.$$

Thus, if we take a finite rank operator $u\colon A \to B$ between C^*-algebras, and if we let $y \in A^* \otimes B$ be associated to u, then $\|u\|_{\mathrm{dec}} \leq \delta(y)$ by Theorem 2.3(iii). The meaning of Theorem 2.1(i) is that, in fact,

$$\|u\|_{\mathrm{dec}} = \delta(y).$$

By Theorem 2.1(ii), the same equality holds if A is a von Neumann algebra and $y \in A_* \otimes B$.

Proof of Theorem 2.1. Let us prove (i). We consider a finite rank bounded operator $u\colon A \to B$, and we take $\varphi_1, \ldots, \varphi_N$ in A^* and $b_1, \ldots, b_N$ in B such that the tensor

$\sum \varphi_k \otimes b_k$ in $A^* \otimes B$ represents u. By part (iii) of Theorem 2.3, it suffices to show that

$$\delta\Big(\sum \varphi_k \otimes b_k\Big) \leq \|u\|_{\text{dec}}. \tag{2.4}$$

We give ourselves a Hilbert space $\mathcal{H}$, a $*$-representation $\pi\colon B \to B(\mathcal{H})$, and a completely contractive map $\sigma\colon A^* \to B(\mathcal{H})$, and we assume that the ranges of π and σ commute. Let $M = \pi(B)' \subset B(\mathcal{H})$ be the commutant of $\pi(B)$; then M is a von Neumann algebra. We may now regard σ as a complete contraction from A^* into M. Applying (2.1) with $R = A^{**}$, we identify σ with an element θ in $A^{**}\overline{\otimes}M$ such that $\|\theta\| \leq 1$. By Kaplansky's density theorem, there exists a net $(\theta_i)_i \subset A^{**}\overline{\otimes}M$ converging to θ in the w^*-topology of $A^{**}\overline{\otimes}M$ such that, for every i, $\|\theta_i\| \leq 1$ and θ_i belongs to the algebraic tensor product $A \otimes M$. Applying (2.1) again (or merely (1.1)), we associate a complete contraction $\sigma_i\colon A^* \to M$ to each θ_i, and looking at (2.2), we observe that the w^*-convergence of θ_i towards θ implies

$$\forall \varphi \in A^*, \quad \sigma_i(\varphi) \xrightarrow{w^*} \sigma(\varphi). \tag{2.5}$$

We now let $Q\colon A \otimes_{\max} M \to A \otimes_{\min} M$ be the canonical $*$-representation induced by the identity on $A \otimes M$, so that

$$\frac{\big(A \otimes_{\max} M\big)}{\ker Q} = A \otimes_{\min} M \quad \text{isometrically.} \tag{2.6}$$

For any $1 \leq k \leq N$, we denote by $\widehat{\varphi}_k\colon A \otimes_{\min} M \to M$ the (unique) bounded extension of $\varphi_k \otimes I_M$. Now let $\widetilde{u}\colon A \otimes_{\max} M \to B \otimes_{\max} M$ be the extension of $u \otimes I_M$ given by Lemma 1.1. For any $a \in A, m \in M$, we have $(u \otimes I_M)(a \otimes m) = \sum_{k=1}^N b_k \otimes (\varphi_k(a)m)$, and hence,

$$\forall \tau \in A \otimes_{\max} M, \quad \widetilde{u}(\tau) = \sum_{k=1}^N b_k \otimes \big(\widehat{\varphi}_k\big(Q(\tau)\big)\big).$$

We deduce that $\widetilde{u}$ vanishes on $\ker Q$, and hence by (2.6), we obtain the following for every i:

$$\Big\| \sum_{k=1}^N b_k \otimes \big(\widehat{\varphi}_k\big(\theta_i\big)\big) \Big\|_{B \otimes_{\max} M} = \big\| \widetilde{u}\big(\theta_i\big) \big\|_{B \otimes_{\max} M} \leq \|\widetilde{u}\| \|\theta_i\|_{A \otimes_{\min} M}.$$

Since $\|\theta_i\|_{A \otimes_{\min} M} \leq 1$, we deduce from Lemma 1.1 that

$$\Big\| \sum_{k=1}^N b_k \otimes \big(\widehat{\varphi}_k\big(\theta_i\big)\big) \Big\|_{B \otimes_{\max} M} \leq \|u\|_{\text{dec}}.$$

Considering the relation between θ_i and σ_i (see (2.2)), we have actually proved

$$\left\| \sum_{k=1}^{N} b_k \otimes \sigma_i\left(\varphi_k\right) \right\|_{B \otimes_{\max} M} \leq \|u\|_{\mathrm{dec}}.$$

The $*$-representation $\pi : B \to B(\mathcal{H})$ is valued in the commutant of M, and hence the linear mapping from $B \otimes M$ into $B(\mathcal{H})$ defined by taking $b \otimes m$ to $\pi(b)m$ for any $b \in B$ and $m \in M$ extends to a $*$-representation from $B \otimes_{\max} M$ into $B(\mathcal{H})$. Consequently,

$$\left\| \sum_{k=1}^{N} \pi\left(b_k\right) \sigma_i\left(\varphi_k\right) \right\|_{B(\mathcal{H})} \leq \|u\|_{\mathrm{dec}}.$$

The multiplication mapping is w^*-continuous on $B(\mathcal{H})$; hence for any $1 \leq k \leq N$, we have

$$\pi\left(b_k\right) \sigma_i\left(\varphi_k\right) \xrightarrow{w^*} \pi(b_k) \sigma(\varphi_k)$$

by (2.5). Thus $\sigma \cdot \pi(\sum \varphi_k \otimes b_k)$ is the w^*-limit of $\sum \pi(b_k) \sigma_i(\varphi_k)$, from which we get

$$\left\| \sigma \cdot \pi \left(\sum \varphi_k \otimes b_k \right) \right\| \leq \|u\|_{\mathrm{dec}}.$$

Taking the supremum over the commuting pairs (σ, π), we obtain, by Definition 2.2, the desired inequality (2.4).

The proof of (ii) is almost identical, using the second part of Theorem 2.3 instead of the third one. We omit it. □

As an application of Remark 2.4, we obtain the following corollary, which is used later in this paper.

COROLLARY 2.6. *Let A, B be two C^*-algebras, let $j : B \to B^{**}$ be the canonical inclusion map, and let $u : A \to B$ be a finite rank bounded operator. Then $\|u\|_{\mathrm{dec}} = \|ju\|_{\mathrm{dec}}$.*

It should be noticed that the equality $\|u\|_{\mathrm{dec}} = \|ju\|_{\mathrm{dec}}$ obtained above does not remain true if we remove the finite rank assumption on u. Indeed, one may find a compact set K, and a bounded operator $u : C(K) \to C(K)$ on the C^*-algebra $C(K)$ of all complex-valued continuous functions on K, which is not decomposable (see [11] and the references therein). However, ju is valued in the commutative von Neumann algebra $C(K)^{**}$ and, hence, is automatically decomposable, with $\|ju\|_{\mathrm{dec}} = \|ju\| = \|u\|$.

We now focus on the situation where A (or B) is a sub-C^*-algebra of $M_N(C)$ for some integer $N \geq 1$ and some commutative C^*-algebra C, and we improve Theorem 2.1 in these cases. We recall that if A or B is commutative, then by Stinespring's theorems, a linear map $u : A \to B$ is positive if and only if it is completely positive (see, e.g., [24]).

Proposition 2.7. *Let $u\colon A \to B$ be a finite rank bounded operator between C^*-algebras, and let $\varepsilon > 0$.*

(i) *Assume that there exist an integer $N \geq 1$ and a commutative C^*-algebra C such that $A \subset M_N(C)$. Then there exist $n \geq 1$ and two linear maps $\alpha\colon A \to \ell_\infty^n(M_N),\ \beta\colon \ell_\infty^n(M_N) \to B$ such that $u = \beta\alpha$ and $\|\alpha\|_{\mathrm{cb}}\|\beta\|_{\mathrm{dec}} \leq (1+\varepsilon)\|u\|_{\mathrm{dec}}$.*

(ii) *Assume that A is a commutative C^*-algebra. Then there exist $\varphi_1, \ldots, \varphi_n$ in A^* and $a_1, \ldots, a_n, b_1, \ldots, b_n$ in B such that for any x in A, we have $u(x) = \sum_k \varphi_k(x) a_k b_k$, and*

$$\sup_k \big\{\|\varphi_k\|\big\} \Big\| \sum_k a_k a_k^* \Big\|^{1/2} \Big\| \sum_k b_k^* b_k \Big\|^{1/2} \leq (1+\varepsilon)\|u\|_{\mathrm{dec}}.$$

Proof. We appeal to a factorization result of Gilles Pisier and the first author in [26]. We assume that $A \subset M_N(C)$, for some commutative C^*-algebra C. By Theorem 2.1, we have $u = \beta\alpha$ with $A \xrightarrow{\alpha} M_k \xrightarrow{\beta} B$ and $\|\alpha\|_{\mathrm{cb}}\|\beta\|_{\mathrm{dec}} \leq (1+\varepsilon)\|u\|_{\mathrm{dec}}$. Further, it follows from Theorem 18 in [26] and its proof that we have a factorization $\alpha = \alpha''\alpha'$, with $A \xrightarrow{\alpha'} \ell_\infty^n(M_N) \xrightarrow{\alpha''} M_k$, and $\|\alpha''\|_{\mathrm{cb}}\|\alpha'\|_{\mathrm{cb}} \leq (1+\varepsilon)\|\alpha\|_{\mathrm{cb}}$. This leads to a new factorization $u = (\beta\alpha'')\alpha'$, which, by (1.6) and (1.5), provides part (i) in Proposition 2.7. Now assume that A is commutative; then (i) holds with $N = 1$. Let $(e_k)_{1\leq k\leq n}$ be the canonical basis of ℓ_∞^n and let $\varphi_k = \alpha^*(e_k^*)$ for all $1 \leq k \leq n$. Then for any $x \in A$, we have $u(x) = \sum_k \varphi_k(x)\beta(e_k)$, and hence (ii) is an immediate consequence of [28, Lemma 1.16]. □

Remark 2.8. Let $N \geq 1$ be an integer, let C be a commutative C^*-algebra, and let $B \subset M_N(C)$ be a sub-C^*-algebra of $M_N(C)$. Arguing as above, and using the fact that B is nuclear, we obtain (without appealing to Theorem 2.1) the following analog of Proposition 2.7, whose proof is left to the reader: For any operator space X, for any finite rank bounded operator $u\colon X \to B$, and for any $\varepsilon > 0$, there exist an integer $n \geq 1$ and two linear maps $\alpha\colon X \to \ell_\infty^n(M_N),\ \beta\colon \ell_\infty^n(M_N) \to B$ such that $u = \beta\alpha$ and $\|\alpha\|_{\mathrm{cb}}\|\beta\|_{\mathrm{cb}} \leq (1+\varepsilon)\|u\|_{\mathrm{cb}}$. Note that since B is nuclear, we have $\|\beta\|_{\mathrm{dec}} = \|\beta\|_{\mathrm{cb}}$ in the above statement. Furthermore, if $X = A$ is a C^*-algebra, then $\|u\|_{\mathrm{dec}} = \|u\|_{\mathrm{cb}}$. This follows from, for example, [29, Section 6.3]; see the next section of the present paper for more on this theme.

We now turn to our application to local reflexivity. By definition (see [5], [7]), an operator space X is said to be locally reflexive provided that, for any finite-dimensional operator space F and any completely bounded map $u\colon F \to X^{**}$, there exists a net of operators $u_i\colon F \to X$ satisfying $\|u_i\|_{\mathrm{cb}} \leq \|u\|_{\mathrm{cb}}$ for any i and such that, for any $f \in F$, $u_i(f)$ converges to $u(f)$ in the w^*-topology of X^{**}. Taking $E = F^*$, we see that X is locally reflexive if and only if

$$\big(X \otimes_{\min} E\big)^{**} = X^{**} \otimes_{\min} E \quad \text{isometrically}$$

for any finite-dimensional operator space E. The next lemma is a useful reformulation of local reflexivity, which goes back to [10]. First note that given any operator space Y, there is a canonical embedding

$$X^{**} \otimes Y \longrightarrow \big(X \otimes_{\min} Y\big)^{**} \tag{2.7}$$

defined as follows. To any $\xi \in (X \otimes_{\min} Y)^*$, we may associate $T\colon Y \to X^*$ by letting $\langle T(y), x\rangle = \langle \xi, x \otimes y\rangle$ for any $y \in Y$ and $x \in X$. Then given x^{**} in X^{**} and $y \in Y$, we can regard $x^{**} \otimes y$ as an element of $(X \otimes_{\min} Y)^{**}$ by letting, for any ξ as above, $\langle x^{**} \otimes y, \xi\rangle = \langle T(y), x^{**}\rangle$. By linearity, this yields (2.7). With the same notation, we note that if we take $z \in X^{**} \otimes Y$, and if we denote by $u\colon X^* \to Y$ the finite rank operator associated to z, then

$$\langle z, \xi\rangle = \mathrm{tr}(uT). \tag{2.8}$$

In the following statement, we consider (2.7) in the case when $Y = B(H)$.

LEMMA 2.9 [10, Section 3]. *Let X be any operator space. Then X is locally reflexive if and only if for any Hilbert space H, we have a contractive (and hence, isometric) embedding*

$$X^{**} \otimes_{\min} B(H) \subset \big(X \otimes_{\min} B(H)\big)^{**}.$$

The following result yields a different, simplified approach to the local reflexivity result of Effros-Junge-Ruan [6].

THEOREM 2.10. *Let R be a von Neumann algebra, let B be a C^*-algebra, and let $u\colon R \to B$ be a finite rank bounded operator.*

(i) *For any $\varepsilon > 0$, there exists a net $u_i\colon R \to B$ of w^*-continuous finite rank operators such that* $\sup_i\{\|u_i\|_{\mathrm{dec}}\} \leq (1+\varepsilon)\|u\|_{\mathrm{dec}}$, *and for any bounded operator $T\colon B \to R$, we have* $\mathrm{tr}(u_i T) \longrightarrow \mathrm{tr}(uT)$.

(ii) *The operator space R_* is locally reflexive.*

Proof. We let $\varepsilon > 0$. By Theorem 2.1, we may find $n \geq 1$, $\alpha\colon R \to M_n$, and $\beta\colon M_n \to B$ such that $u = \beta\alpha$ and $\|\alpha\|_{\mathrm{cb}}\|\beta\|_{\mathrm{dec}} \leq (1+\varepsilon)\|u\|_{\mathrm{dec}}$. In the identification $CB(R, M_n) = M_n(R^*)$ given by (1.1), the subspace $M_n(R_*) \subset M_n(R^*)$ corresponds to the space of w^*-continuous maps from R into M_n. Since $M_n(R_*)^{**} = M_n(R^*)$ isometrically (see [1]), it therefore follows from Goldstine's lemma that there exists a net $\alpha_i\colon R \to M_n$ of w^*-continuous maps with $\|\alpha_i\|_{\mathrm{cb}} \leq \|\alpha\|_{\mathrm{cb}}$ and

$$\forall r \in R, \quad \alpha_i(r) \longrightarrow \alpha(r). \tag{2.9}$$

We let $u_i = \beta\alpha_i$ and consider a bounded operator $T\colon B \to R$. We denote by $(E_{pq})_{1\leq p,q\leq n}$ and $(E^*_{pq})_{1\leq p,q\leq n}$ the canonical basis of M_n and its dual basis, respectively. Then we have

$$\begin{aligned}\mathrm{tr}(u_i T) = \mathrm{tr}\big(\alpha_i T\beta\big) &= \sum_{1\le p,q\le n} \big\langle \alpha_i\big((T\beta)\big(E_{pq}\big)\big), E_{pq}^*\big\rangle \\ &\longrightarrow \sum_{1\le p,q\le n} \Big\langle \alpha\big((T\beta)\big(E_{pq}\big)\big), E_{pq}^*\Big\rangle \quad \text{by (2.9)} \\ &= \mathrm{tr}(\alpha T\beta) = \mathrm{tr}(uT).\end{aligned}$$

Since $\|u_i\|_{\mathrm{dec}} \le (1+\varepsilon)\|u\|_{\mathrm{dec}}$ (see Remark 2.5), this concludes the proof of part (i).

Let us now deduce that R_* is locally reflexive. We fix a Hilbert space H and we take z in $R^*\otimes B(H)$. We denote by $u\colon R\to B(H)$ the finite rank operator associated to z and we give ourselves some $\varepsilon>0$. Then we consider $\xi\in(R_*\otimes_{\min} B(H))^*$ and we let $T\colon B(H)\to R$ be the linear mapping associated to ξ. By (1.5), $\|u\|_{\mathrm{cb}}=\|u\|_{\mathrm{dec}}$; hence we can find, by the first part of the theorem, a net $u_i\colon R\to B(H)$ of w^*-continuous finite rank linear maps such that $\|u_i\|_{\mathrm{cb}}\le(1+\varepsilon)\|u\|_{\mathrm{cb}}$ and $\mathrm{tr}(u_iT)\longrightarrow\mathrm{tr}(uT)$. Each u_i is represented by some $z_i\in R_*\otimes B(H)$, and by (2.8) we have

$$\big|\,\mathrm{tr}\big(u_i T\big)\big| = |\langle z_i,\xi\rangle| \le \|u_i\|_{\mathrm{cb}}\|\xi\|_{(R_*\otimes_{\min}B(H))^*} \le (1+\varepsilon)\|u\|_{\mathrm{cb}}\|\xi\|_{(R_*\otimes_{\min}B(H))^*}.$$

Passing to the limit, we get

$$\begin{aligned}|\langle z,\xi\rangle| = |\,\mathrm{tr}(uT)| &\le (1+\varepsilon)\|u\|_{\mathrm{cb}}\|\xi\|_{(R_*\otimes_{\min}B(H))^*} \\ &= (1+\varepsilon)\|z\|_{R^*\otimes_{\min}B(H)}\|\xi\|_{(R_*\otimes_{\min}B(H))^*}.\end{aligned}$$

Since ξ and $\varepsilon>0$ are arbitrary, we obtain that $\|z\|_{(R_*\otimes_{\min}B(H))^{**}}\le\|z\|_{R^*\otimes_{\min}B(H)}$. By Lemma 2.9, this proves the result. □

3. Factorization through M_n with respect to the completely bounded norm. In this section we are primarily interested in the problem of whether Theorem 2.1 remains valid with the completely bounded norm instead of the decomposable one. In order to investigate this problem and related ones, it is convenient to introduce the following.

Definition 3.1. Let $u\colon X\to Y$ be a finite rank bounded operator between operator spaces. We set

$$\gamma(u)=\inf\{\|\alpha\|_{\mathrm{cb}}\|\beta\|_{\mathrm{cb}}\},$$

where the infimum runs over all $n\ge1$ and all $\alpha\colon X\to M_n,\ \beta\colon M_n\to Y$ such that $u=\beta\alpha$.

It is well known and easy to check that for any two operator spaces X,Y, $u\mapsto\gamma(u)$ is a norm on the space of finite rank operators from X into Y. Now let A,B be two C^*-algebras. Given any finite rank $u\colon A\to B$, we have, by Theorem 2.1,

$$\|u\|_{\mathrm{cb}}\le\gamma(u)\le\|u\|_{\mathrm{dec}}. \tag{3.1}$$

This leads to the following two natural questions: For which C^*-algebras B do we have $\|u\|_{\mathrm{cb}}=\gamma(u)$ for any C^*-algebra A and any finite rank $u\colon A\to B$? For which

C^*-algebras B do we have $\|u\|_{\mathrm{cb}} = \|u\|_{\mathrm{dec}}$ for any C^*-algebra A and any finite rank $u\colon A \to B$? In Propositions 3.2 and 3.3, we provide some partial answers.

PROPOSITION 3.2. *Let B be a von Neumann algebra. Then B is injective if and only if, for any C^*-algebra A and any finite rank linear map $u\colon A \to B$, we have $\|u\|_{\mathrm{cb}} = \gamma(u)$.*

Proof. Assume that B is injective. Then for any $u\colon A \to B$, we have $\|u\|_{\mathrm{cb}} = \|u\|_{\mathrm{dec}}$ by (1.5). Hence if u has finite rank, we actually have $\|u\|_{\mathrm{cb}} = \gamma(u)$ by Theorem 2.1.

We now turn to the proof of the "if" part. It is based on a characterization of injectivity of Pisier [27]. We assume that $\|u\|_{\mathrm{cb}} = \gamma(u)$ for any A and any finite rank $u\colon A \to B$. Given an integer $N \geq 1$, we let $C_N \subset M_N$ (resp., $R_N \subset M_N$) be the operator space formed by all matrices with entries equal to zero except in the first column (resp., row). They are both operator spaces isometric to the N-dimensional Hilbertian space ℓ_2^N. Then we denote by $R_N \cap C_N \subset R_N \overset{\infty}{\oplus} C_N$ the operator space structure on ℓ_2^N induced by the mapping $x \in \ell_2^N \mapsto (x, x) \in \ell_2^N \oplus \ell_2^N$ (see, e.g., [27, Section 2] for details).

We give ourselves an operator $\tau\colon R_N \cap C_N \to B$. Let $\mathbb{F}$ denote the free group on infinitely many generators and let $A = C_\lambda^*(\mathbb{F})$ be the reduced C^*-algebra of $\mathbb{F}$. By [13, Proposition 1.3], there exist two linear mappings $w\colon R_N \cap C_N \to A$ and $v\colon A \to R_N \cap C_N$ such that $\|v\|_{\mathrm{cb}}\|w\|_{\mathrm{cb}} \leq 2$ and vw equals the identity on $R_N \cap C_N$. Applying our assumption to $u = \tau v$, we find an integer $n \geq 1$ and a factorization $u = \beta\alpha$ with $A \xrightarrow{\alpha} M_n \xrightarrow{\beta} B$ and

$$\|\alpha\|_{\mathrm{cb}}\|\beta\|_{\mathrm{cb}} \leq 2\|u\|_{\mathrm{cb}} \leq 2\|\tau\|_{\mathrm{cb}}\|v\|_{\mathrm{cb}}. \tag{3.2}$$

We now let $\sigma = \alpha w\colon R_N \cap C_N \to M_n$. By means of the completely isometric embedding $R_N \cap C_N \subset R_N \overset{\infty}{\oplus} C_N$, it is easy to derive from Wittstock's extension theorem [33] that there exists a decomposition $\sigma = \sigma_1 + \sigma_2$, with

$$\|\sigma_1\colon R_N \to M_n\|_{\mathrm{cb}} \leq \|\sigma\|_{\mathrm{cb}} \qquad \text{and} \qquad \|\sigma_2\colon C_N \to M_n\|_{\mathrm{cb}} \leq \|\sigma\|_{\mathrm{cb}}. \tag{3.3}$$

For $k = 1, 2$, we let $\tau_k = \beta\sigma_k$. Then we have

$$\tau_1 + \tau_2 = \beta\sigma = \beta\alpha w = uw = \tau v w = \tau.$$

Furthermore,

$$\begin{aligned}\|\tau_1\colon R_N \to B\|_{\mathrm{cb}} &\leq \|\beta\|_{\mathrm{cb}}\|\sigma\|_{\mathrm{cb}} \quad \text{by (3.3)}\\ &\leq \|\beta\|_{\mathrm{cb}}\|\alpha\|_{\mathrm{cb}}\|w\|_{\mathrm{cb}}\\ &\leq 2\|\tau\|_{\mathrm{cb}}\|v\|_{\mathrm{cb}}\|w\|_{\mathrm{cb}} \quad \text{by (3.2)}\\ &\leq 4\|\tau\|_{\mathrm{cb}}.\end{aligned}$$

Similarly, $\|\tau_2\colon C_N \to B\|_{\mathrm{cb}} \leq 4\|\tau\|_{\mathrm{cb}}$. By [27, Theorem 2.9], that decomposition result on τ shows that B is injective. □

We recall that by definition (refer to [18]), a C^*-algebra B has the weak expectation property (WEP) provided that the canonical inclusion map of B into B^{**} admits a factorization

$$B \xrightarrow{J} B(H) \xrightarrow{P} B^{**} \tag{3.4}$$

for some completely positive contractions J and P. The class of C^*-algebras with the WEP includes nuclear C^*-algebras and injective von Neumann algebras.

PROPOSITION 3.3. *Let A and B be two C^*-algebras and assume that B has the WEP. Let $u\colon A \to B$ be any finite rank linear map. Then we have $\|u\|_{\mathrm{cb}} = \gamma(u) = \|u\|_{\mathrm{dec}}$.*

Proof. Let $j\colon B \to B^{**}$ be the canonical inclusion map and let P and J be as in (3.4) such that $j = PJ$. We give ourselves a finite rank bounded operator $u\colon A \to B$. We have $ju = PJu$, and hence by (1.6) and (1.5), we obtain that $\|ju\|_{\mathrm{dec}} \leq \|P\|_{\mathrm{dec}}\|Ju\|_{\mathrm{cb}}$. Since P is a completely positive contraction, we have $\|P\|_{\mathrm{dec}} \leq 1$ by (1.4). On the other hand, $\|J\|_{\mathrm{cb}} \leq 1$, and hence we deduce that $\|ju\|_{\mathrm{dec}} \leq \|u\|_{\mathrm{cb}}$. Appealing to Corollary 2.6, we thus obtain $\|u\|_{\mathrm{dec}} \leq \|u\|_{\mathrm{cb}}$, from which we get the result by (3.1). □

The previous result has an operator space interpretation in terms of the Haagerup tensor product. We refer the reader to [2], [8], [29] for the definition and some background on this tensor product that we denote by $\overset{h}{\otimes}$. Let E be an operator space and let B be a C^*-algebra. It is easy to check that the linear mapping from $B \otimes E \otimes B$ into $E \otimes B$ that takes $a \otimes e \otimes b$ to $e \otimes ab$ extends to a complete contraction $Q\colon B \overset{h}{\otimes} E \overset{h}{\otimes} B \to E \otimes_{\min} B$. Moreover, it follows from Theorem 2.3(i) that Q is a quotient mapping if and only if $E \otimes_{\min} B = E \overset{\delta}{\otimes} B$ isometrically.

COROLLARY 3.4. *Let R be a von Neumann algebra and let B be a C^*-algebra with the WEP. Then the complete contraction*

$$Q\colon B \overset{h}{\otimes} R_* \overset{h}{\otimes} B \longrightarrow R_* \otimes_{\min} B$$

defined by $Q(a \otimes \varphi \otimes b) = \varphi \otimes ab$ for $a, b \in B$ and $\varphi \in R_*$ is a complete quotient mapping. That is, for any $n \geq 1$, $I_{M_n} \otimes Q$ is a quotient mapping from $M_n(B \overset{h}{\otimes} R_* \overset{h}{\otimes} B)$ onto $M_n(R_* \otimes_{\min} B)$.

Proof. Under our assumption, we have

$$R_* \otimes_{\min} B = R_* \overset{\delta}{\otimes} B.$$

Indeed, let $y \in R_* \otimes B$ and let $u\colon R \to B$ be the linear map represented by y. Then $\|u\|_{\mathrm{dec}} = \delta(y)$ by Theorem 2.1(ii) (see also Remark 2.5), whereas $\|u\|_{\mathrm{cb}} = \|y\|_{\min}$

by (1.1), from which we get $\delta(y) = \|y\|_{\min}$ by Proposition 3.3. Thus Q is a quotient mapping.

Given any $n \geq 1$, the C^*-algebra $M_n(B)$ has the WEP. Thus, taking into account the isometric identity $R_* \otimes_{\min} M_n(B) = M_n(R_* \otimes_{\min} B)$, the previous argument yields a quotient mapping

$$Q_n : M_n(B) \overset{h}{\otimes} R_* \overset{h}{\otimes} M_n(B) \longrightarrow M_n\big(R_* \otimes_{\min} B\big).$$

Let $p_n : M_n \otimes M_n \to M_n$ be the multiplication mapping on M_n and let $Z = B \overset{h}{\otimes} R_* \overset{h}{\otimes} B$. Via the identification $M_n(B) \overset{h}{\otimes} R_* \overset{h}{\otimes} M_n(B) = M_n \otimes Z \otimes M_n$, we may consider

$$I_Z \otimes p_n : M_n(B) \overset{h}{\otimes} R_* \overset{h}{\otimes} M_n(B) \longrightarrow M_n\Big(B \overset{h}{\otimes} R_* \overset{h}{\otimes} B\Big),$$

and it is clear that $Q_n = (I_{M_n} \otimes Q)(I_Z \otimes p_n)$. To prove that $I_{M_n} \otimes Q$ is a quotient mapping, it suffices to see that $I_Z \otimes p_n$ is a contraction. This can be checked very easily by means of elementary manipulations on the Haagerup tensor product; we skip the details. □

Uffe Haagerup informed us recently that he could prove a converse to Proposition 3.3, which generalizes the main result of [11]. Let B be a C^*-algebra. Haagerup proved that if for any $n \geq 1$ and any $u : \ell_\infty^n \to B$, we have $\|u\|_{\mathrm{dec}} = \|u\|_{\mathrm{cb}}$, then B has the WEP. In Proposition 3.5, we give an alternate characterization of the WEP. We recall that we denote by $\mathcal{K}$ the C^*-algebra of all compact operators on the separable Hilbert space ℓ_2.

Proposition 3.5. *For any C^*-algebra B, the following assertions are equivalent.*
(i) *B has the WEP.*
(ii) *For any linear map $u : \ell_\infty^3 \to \mathcal{K} \otimes_{\min} B$, we have $\|u\|_{\mathrm{dec}} = \|u\|_{\mathrm{cb}}$.*

Proof. When a C^*-algebra B has the WEP, then $\mathcal{K} \otimes_{\min} B$ has the WEP as well, and hence (i) implies (ii) by Proposition 3.3.

We now assume (ii). Let $\mathbb{F}_2$ be the free group with two generators g_1, g_2, and let $C^*(\mathbb{F}_2)$ denote the full C^*-algebra of $\mathbb{F}_2$. By Kirchberg's work [17], (i) is equivalent to

$$C^*(\mathbb{F}_2) \otimes_{\min} B = C^*(\mathbb{F}_2) \otimes_{\max} B. \tag{3.5}$$

We make use of Pisier's approach to that result in [28]. By a standard argument, we can assume that B is unital. For $i = 1, 2$, let $U_i \in C^*(\mathbb{F}_2)$ be the unitary associated to g_i. Then we let $E \subset C^*(\mathbb{F}_2)$ be the linear span of the unit of $C^*(\mathbb{F}_2)$ and of $\{U_1, U_2\}$. By [28, Theorem 1.1], (3.5) holds if and only if

$$E \otimes_{\min} B \subset C^*(\mathbb{F}_2) \otimes_{\max} B \quad \text{completely isometrically.} \tag{3.6}$$

Let $n \geq 1$ be an integer and let $y \in M_n(E \otimes_{\min} B)$ be given. By [25, Section 3], the operator space E is completely isometric to $\ell_1^3 = (\ell_\infty^3)^*$, and moreover, $M_n(E \otimes_{\min} B) = E \otimes_{\min} M_n(B)$ isometrically, from which we get $M_n(E \otimes_{\min} B) = CB(\ell_\infty^3, M_n(B))$. Let $u\colon \ell_\infty^3 \to M_n(B)$ be the linear mapping corresponding to y in this identification. Then by our assumption (ii), we have $\|u\|_{\rm cb} = \|u\|_{\rm dec}$. On the other hand, $\|u\|_{\rm dec} = \|y\|_{C^*(\mathbb{F}_2) \otimes_{\max} M_n(B)}$ by [28, Lemma 1.20]. Since $C^*(\mathbb{F}_2) \otimes_{\max} M_n(B) = M_n(C^*(\mathbb{F}_2) \otimes_{\max} B)$, we finally obtain $\|y\|_{M_n(E \otimes_{\min} B)} = \|y\|_{M_n(C^*(\mathbb{F}_2) \otimes_{\max} B)}$. This proves (3.6) and concludes the proof. □

Remark 3.6. We wish to mention yet one more characterization of the WEP, also based on factorization through matrix spaces. Let B be a C^*-algebra. Then B has the WEP if and only if, for any pair of finite-dimensional operator spaces X, Y and for every linear map $u\colon X \to Y$ that factors through B, we have

$$\gamma(u) = \inf\left\{\|w\|_{\rm cb}\|v\|_{\rm cb}\right\},$$

where the infimum in the right-hand side runs over all $X \stackrel{v}{\longrightarrow} B \stackrel{w}{\longrightarrow} Y$ such that $u = wv$. According to [6, Theorem 7.7], this property holds if B has the WEP.

Let us prove the converse. We give ourselves a Hilbert space H such that B is contained in $B(H)$ as a C^*-algebra. We consider the net I of pairs (X, Y), where X is a finite-dimensional subspace of B and Y is a finite-dimensional quotient space of B. (Equivalently, by (1.3), Y^* is a finite-dimensional subspace of B^*.) The canonical order is given by $(X_0, Y_0) \leq (X_1, Y_1)$ if and only if $X_0 \subset X_1$ and $Y_0^* \subset Y_1^*$. Given such a pair $i = (X, Y) \in I$, we denote by $j_i\colon X \to B$ the inclusion map and by $q_i\colon B \to Y$ the quotient map. According to our assumption, there exist a factorization $q_i j_i = \beta_i \alpha_i$, with $X \stackrel{\alpha_i}{\longrightarrow} M_{n_i} \stackrel{\beta_i}{\longrightarrow} Y$, such that $\|\alpha_i\|_{\rm cb} \leq 1$ and $\|\beta_i\|_{\rm cb} \leq 1 + \dim(Y)^{-1}$. Using Wittstock's extension theorem (see [33]), we can extend α_i to a complete contraction $\widehat{\alpha_i}\colon B(H) \to M_{n_i}$. Now let $M = \ell_\infty\{I; M_{n_i}\}$ be the C^*-algebraic direct sum of the M_{n_i}'s; then we obtain a complete contraction $W\colon B(H) \to M$ by letting $W(b) = (\widehat{\alpha_i}(b))_{i \in I}$. Let $\mathcal{U}$ be an ultrafilter refining the canonical order on I. We can define a complete contraction $T\colon B^* \to M^*$ by letting

$$\left\langle T(f), (z_i)_{i \in I}\right\rangle = \lim_{i=(X,Y)\in\mathcal{U},\, f \in Y^*} \left\langle \beta_i^*(f), z_i\right\rangle,$$

for any $f \in B^*$ and any $(z_i)_{i \in I} \in M = \ell_\infty\{I; M_{n_i}\}$. By construction, we obviously have

$$\forall b \in B,\ f \in B^*, \quad \langle f, b\rangle = \left\langle T(f), W(b)\right\rangle.$$

This shows that $(W_{/B})^* T$ equals the identity on B^*. Consequently, $T^* W^{**}\colon B(H)^{**} \to B^{**}$ is a projection from $B(H)^{**}$ onto its sub-C^*-algebra B^{**}. Since $T^* W^{**}$ is a complete contraction, we deduce from Tomiyama's theorem that $T^* W^{**}$ is completely positive. Restricting to $B(H)$, we obtain that $T^* W\colon B(H) \to B^{**}$ is a completely positive contraction whose restriction to B is the canonical inclusion of B into B^{**}. This yields the assertion that B has the WEP.

We now turn to an application of Proposition 3.3 to the theory of completely integral maps on operator spaces. We briefly recall the relevant definitions, which were introduced in [9], [10]. Let X, Y be two operator spaces and let $T\colon X \to Y$ be a completely bounded map. We let $\Phi\colon X^* \widehat{\otimes} Y \to CB(X,Y)$ be the contractive mapping extending the canonical embedding $X^* \otimes Y \subset CB(X,Y)$. Then T is said to be completely nuclear if it lies in the range of Φ, and its completely nuclear norm $\nu^\circ(T)$ is defined as the quotient norm of T in $(X^* \widehat{\otimes} Y)/\ker\Phi$. Next, T is said to be completely integral if it is the point-norm limit of a bounded net $(T_i)_i$ of completely nuclear maps from X into Y, and its completely integral norm $\iota^\circ(T)$ is defined as the infimum of $\sup_i\{\nu^\circ(T_i)\}$ over all such possible nets. Lastly, T is said to be completely 1-summing if there is a constant $C > 0$ such that for any $n \geq 1$,

$$\big\| I_{S_1^n} \otimes T\colon S_1^n \otimes_{\min} X \longrightarrow S_1^n \widehat{\otimes} Y \big\| \leq C. \tag{3.7}$$

In that case, its completely 1-summing norm $\pi_1^\circ(T)$ is defined as the smallest C satisfying (3.7). The space of all completely nuclear maps (resp., completely integral maps, resp., completely 1-summing maps) from X into Y is denoted by $N(X,Y)$ (resp., $I(X,Y)$, resp., $\Pi_1(X,Y)$). In general, the following contractive inclusions hold:

$$N(X,Y) \subset I(X,Y) \subset \Pi_1(X,Y).$$

We refer the reader to [9], [10], [14], [30] for these inclusions and more information on this topic.

Corollary 3.7. *Let A and B be two C^*-algebras and assume that B has the WEP. Then $I(B,A) = \Pi_1(B,A)$, and for any completely 1-summing $T\colon B \to A$,*

$$\pi_1^\circ(T) = i^\circ(T).$$

Proof. Let $T\colon B \to A$ be completely 1-summing. Given two arbitrary finite families $(b_k)_{1\leq k\leq N}$ in B and $(\varphi_k)_{1\leq k\leq N}$ in A^*, we check that

$$\Big|\sum_{k=1}^{N} \big\langle T(b_k), \varphi_k \big\rangle\Big| \leq \pi_1^\circ(T) \, \Big\|\sum_{k=1}^{N} \varphi_k \otimes b_k\Big\|_{A^*\otimes_{\min} B}. \tag{3.8}$$

By [10] (see also [6, Corollary 4.3]), this shows that T is completely integral with $i^\circ(T) \leq \pi_1^\circ(T)$, which yields the result.

We may assume that $\|\sum \varphi_k \otimes b_k\|_{A^*\otimes_{\min} B} < 1$. By (1.1), the linear map $u\colon A \to B$ associated to $\sum \varphi_k \otimes b_k$ satisfies $\|u\|_{\mathrm{cb}} = \|\sum \varphi_k \otimes b_k\|_{A^*\otimes_{\min} B}$, and hence by Proposition 3.3, we have $\gamma(u) < 1$. Hence there exists a factorization $u = \beta\alpha$, for some $\alpha\colon A \to M_n,\ \beta\colon M_n \to B$ satisfying $\|\alpha\|_{\mathrm{cb}}\|\beta\|_{\mathrm{cb}} < 1$. Then we have

$$\Big|\sum_{k=1}^{N} \big\langle T(b_k), \varphi_k \big\rangle\Big| \leq \big|\mathrm{tr}(Tu)\big| = \big|\mathrm{tr}(T\beta\alpha)\big| = \big|\mathrm{tr}(\alpha T\beta)\big|.$$

Let $z \in S_1^n \otimes B$ represent β. Then the tensor $(I_{S_1^n} \otimes T)(z)$ represents the operator $T\beta$ from which, by (3.7), we get

$$\begin{aligned}\left|\sum_{k=1}^{N} \langle T(b_k), \varphi_k \rangle\right| &\leq \|\alpha\|_{\rm cb} \big\| \big(I_{S_1^n} \otimes T\big)(z)\big\|_{S_1^n \widehat{\otimes} A} \\ &\leq \|\alpha\|_{\rm cb} \pi_1^{\circ}(T) \|z\|_{S_1^n \otimes_{\min} B} = \|\alpha\|_{\rm cb} \pi_1^{\circ}(T) \|\beta\|_{\rm cb} \\ &< \pi_1^{\circ}(T).\end{aligned}$$

This proves (3.8) and concludes the proof. □

Remark 3.8. Let B be a nuclear C^*-algebra. Then for any operator space Y, any finite rank operator $u \colon Y \to B$ satisfies $\|u\|_{\rm cb} = \gamma(u)$ (see, e.g., [29, Section 6.3]). Hence the proof of Corollary 3.7 actually yields

$$\text{for any operator space } Y, \quad I(B, Y) = \Pi_1(B, Y) \quad \text{isometrically.} \tag{3.9}$$

Let us now focus on the (nuclear) C^*-algebra $\mathcal{K}$. Recall that $S_1 = \mathcal{K}^*$ completely isometrically. We observe that for any Y, we have $N(\mathcal{K}, Y) = I(\mathcal{K}, Y)$ isometrically. Indeed, let $j \colon Y \to Y^{**}$ be the canonical inclusion map; then the mapping $T \mapsto jT$ induces an isometric embedding of $I(\mathcal{K}, Y)$ into $(\mathcal{K} \otimes_{\min} Y^*)^*$ by [6, Corollary 4.3]. According to [7, Proposition 4.1], we have $(\mathcal{K} \otimes_{\min} Y^*)^* = S_1 \widehat{\otimes} Y^{**}$ whence $I(\mathcal{K}, Y) = S_1 \widehat{\otimes} Y$ or, equivalently, $I(\mathcal{K}, Y) = N(\mathcal{K}, Y)$. Putting together with (3.9), we obtain that

$$\text{for any operator space } Y, \quad N(\mathcal{K}, Y) = \Pi_1(\mathcal{K}, Y) \quad \text{isometrically.}$$

It should be interesting to characterize the operator spaces X satisfying $N(X, Y) = \Pi_1(X, Y)$ for any Y. We refer to [19] for the corresponding problem in Banach space theory.

We can however mention here a related result. Let B be any C^*-algebra; then

$$B \text{ is nuclear } \Longleftrightarrow N(B, Y) = \Pi_1(B, Y) \text{ isometrically for any finite-dimensional } Y.$$

Indeed, assume that B is nuclear; then it is locally reflexive (see [5]). Hence for any finite-dimensional operator space Y, we have $N(B, Y) = I(B, Y)$ by [10, Theorem 3.6], from which we get the result by (3.9). Conversely, assume that $N(B, Y) = \Pi_1(B, Y)$ for any finite-dimensional Y. Recall that $N(B, Y) = B^* \widehat{\otimes} Y$ for any finite-dimensional Y. Note also that if we consider two arbitrary (possibly infinite-dimensional) operator spaces Y, Z such that $Z \subset Y$ completely isometrically, then we get $\Pi_1(B, Z) \subset \Pi_1(B, Y)$ isometrically. We thus deduce that for any $Z \subset Y$, we have $B^* \widehat{\otimes} Z \subset B^* \widehat{\otimes} Y$ isometrically. Indeed, it is easy to reduce to the case where Y and Z are finite dimensional. By the duality relation (1.2), we can deduce that B^{**} is injective (see [14, Remark 3.5.2] for details). By [3], this yields the result that B is nuclear.

We end this section with an easy consequence of Theorem 2.1, in terms of factorization through finite-dimensional exact operator spaces (in the sense of [26]). It should be compared with [29, Theorem 6.4.11].

COROLLARY 3.9. *Let A be a C^*-algebra and let F be an operator space. Then for any finite rank linear map $u: A \to F$, and for any $\varepsilon > 0$, there exist, for some $n \geq 1$, a subspace $G \subset M_n$ and two linear maps $\alpha: A \to G$, $\beta: G \to F$, such that $u = \beta\alpha$ and $\|\alpha\|_{\rm cb}\|\beta\|_{\rm cb} \leq (1+\varepsilon)\|u\|_{\rm cb}$.*

Proof. Let $J: F \to B(H)$ be a complete isometry. Then by Theorem 2.1, we obtain a factorization $Ju = \beta_1\alpha_1$, with $A \xrightarrow{\alpha_1} M_n \xrightarrow{\beta_1} B(H)$ and $\|\alpha_1\|_{\rm cb}\|\beta_1\|_{\rm cb} \leq (1+\varepsilon)\|u\|_{\rm cb}$. Now take $G = \alpha_1(A)$, let α be induced by α_1, and let β be the restriction of β_1 to G. Then β_1 is valued in F, from which we get the result in Corollary 3.9. □

We observe that the latter result may also be viewed as a consequence of [6, Section 5]. Indeed, it can be obtained as a simple combination of Lemma 5.2, Theorem 5.5, and the duality formula (5.8) in [6].

4. On finite rank operators defined on spaces of Hankel operators. In this section we are concerned with either scalar or vectorial Hankel operators, with respect to a fixed orthonormal basis $(e_i)_{i\geq 1}$ of ℓ_2. We use well-known descriptions of such operators ([20], [21] and [29, Section 8.1]), which require some background on function spaces.

We let $\mathbf{T} = \{z \in \mathbb{C} : |z| = 1\}$ be the scalar unitary group, equipped with its normalized Haar measure. For any $p \in [1, +\infty]$ and any Banach space X, we denote by $L_p(X)$ the Lebesgue-Bochner space $L_p(\mathbf{T}; X)$ of all measurable and p-integrable functions from $\mathbf{T}$ into X. Given any $f \in L_p(X)$ and any $n \in \mathbb{Z}$, we let $\widehat{f}(n) = \int_{\mathbf{T}} f(\xi)\xi^{-n} d\xi$ denote the nth Fourier coefficient of f. Next we can introduce the corresponding Hardy space

$$H_p(X) = \big\{ f \in L_p(X) : \forall n < 0, \widehat{f}(n) = 0 \big\}.$$

For convenience, the *scalar-valued* spaces $L_p(\mathbb{C})$ and $H_p(\mathbb{C})$ are simply denoted by L_p and H_p, respectively.

Recall that L_1 can be canonically regarded as an operator space since it is the predual of the von Neumann algebra L_∞. If X is an operator space, then $L_1(X)$ is regarded as the operator space $L_1 \widehat{\otimes} X$. Similarly, $H_1(X)$ is regarded as the operator space defined by the inclusion $H_1(X) \subset L_1(X)$. See [30, Section 2] for more details about this.

By definition, we say that $T \in B(\ell_2)$ is a Hankel operator if there exists a scalar sequence $(a_q)_{q\geq 1}$ such that $t_{ij} = a_{i+j}$ for any $i, j \geq 1$, where $(t_{ij})_{i,j\geq 1}$ is the matrix of T defined by $t_{ij} = \langle T(e_j), e_i\rangle$. We denote by

$$Ha \subset B(\ell_2)$$

the operator space formed by all Hankel operators. It follows from classical theorems of Nehari [20] and Page [21] that

$$Ha = \frac{L_\infty}{H_\infty} \quad \text{completely isometrically,} \tag{4.1}$$

when the latter is equipped with the quotient operator space structure. Indeed, for any $f \in L_\infty$, there is a (unique) $T = T_f \in Ha$ whose matrix is given by $t_{ij} = \widehat{f}(1-i-j)$ for any $i, j \geq 1$, and the resulting mapping $f \mapsto T_f$ induces an isometry from L_∞/H_∞ onto Ha (refer to [20]). The fact that this isometry is a complete one, which means that $M_n(Ha) = M_n(L_\infty)/M_n(H_\infty)$ isometrically for all $n \geq 1$, is just a reformulation of the finite-dimensional vectorial version of Nehari's theorem due to Page [21].

We now turn to general vectorial Hankel operators. Let H be a Hilbert space. To any $T \in B(\ell_2(H))$, one can associate a matrix $(T_{ij})_{i,j\geq 1}$ with entries in $B(H)$ defined, for any h, k in H, by $\langle T_{ij}(k), h\rangle = \langle T(e_j \otimes k), e_i \otimes h\rangle$. Then we say that $T \in B(\ell_2(H))$ is a $B(H)$-valued Hankel operator if there exists a sequence $(A_q)_{q\geq 1} \subset B(H)$ such that $T_{ij} = A_{i+j}$ for any $i, j \geq 1$. Now let $R \subset B(H)$ be a von Neumann algebra on H. If T is a $B(H)$-valued Hankel operator whose “coefficients” A_q belong to R, we say that T is an R-valued Hankel operator. The operator space of R-valued Hankel operators is denoted by

$$Ha(R) \subset B\big(\ell_2(H)\big).$$

Taking into account the classical duality $H_1^* = L_\infty/H_\infty$ and the complete isometry (4.1), we may state the following lemma, which is a reformulation of some results from [12], which, in turn, rely on Parrot's approach to Nehari's theorem [22].

LEMMA 4.1. *Let R be any von Neumann algebra. Then*

(i) $H_1(R_*) = H_1 \widehat{\otimes} R_*$ *isometrically;*
(ii) $(H_1(R_*))^* = Ha(R)$ *isometrically.*

Proof. We let H be a Hilbert space. Recall that we have $L_1(S_1(H)) = L_1 \widehat{\otimes} S_1(H)$. Moreover, for any operator space E, and any subspace $F \subset E$, the canonical embedding $F \widehat{\otimes} S_1(H) \subset E \widehat{\otimes} S_1(H)$ is isometric (see, e.g., [30, Corollary 1.2]). With $E = L_1$ and $F = H_1$, we derive

$$H_1\big(S_1(H)\big) = H_1 \widehat{\otimes} S_1(H). \tag{4.2}$$

Let $R \subset B(H)$ and let $q\colon S_1(H) \to R_*$ be the quotient mapping defined as the preadjoint map of the canonical embedding $R \to B(H)$. Let $R_\perp = \ker q$ be the preannihilator of R into $S_1(H)$. Then, according to (1.3), we have the completely isometric identification

$$\frac{S_1(H)}{R_\perp} = R_*,$$

if $S_1(H)/R_\perp$ is endowed with the quotient operator space structure. It therefore follows from the functorial properties of $\widehat{\otimes}$ and (4.2) that $I_{H_1} \otimes q : H_1 \otimes S_1(H) \to H_1 \otimes R_*$ extends to a quotient mapping from $H_1(S_1(H))$ onto $H_1 \widehat{\otimes} R_*$. By [12, Theorem 2.2], this yields (i). Furthermore, it follows from above and the proof of Theorem 2.2 in [12] that $(H_1 \widehat{\otimes} R_*)^* = Ha(R)$, from which we get (ii). □

THEOREM 4.2. *Let R be a von Neumann algebra and let H be a Hilbert space.*

(i) *We have $H_1(R_*) \overset{\delta}{\otimes} B(H) = H_1(R_*) \otimes_{\min} B(H)$ isometrically.*

(ii) *For any finite rank bounded operator $u : Ha(R) \to B(H)$, there exist a net of finite rank operators $u_i : Ha(R) \to B(H)$ such that $\gamma(u_i) \le \|u\|_{\rm cb}$ for all i, and*

$$\forall y \in Ha(R), \quad \|u_i(y) - u(y)\| \longrightarrow 0.$$

Remark 4.3. The second point in Theorem 4.2 is equivalent to saying that given a von Neumann algebra R, any finite rank bounded operator defined on $Ha(R)$ is exact (see [29, Section 6.4] or [14, 3.1.3.5] for the definition). This result had been previously obtained (by a different method) by Junge in the *scalar* case (i.e., $R = \mathbb{C}$) in [14, Proposition 3.5.9].

Proof of Theorem 4.2. We give ourselves a Hilbert space $\mathcal{H}$ as well as a completely contractive map $\sigma : H_1(R_*) \to B(\mathcal{H})$ and a $*$-representation $\pi : B(H) \to B(\mathcal{H})$. We assume that σ and π have commuting ranges and let $M = \pi(B(H))'$. We thus have $\sigma \in CB(H_1(R_*), M)$, with $\|\sigma\|_{\rm cb} \le 1$. We claim that

(4.3) there exists $\widehat{\sigma} \in CB(L_1(R_*), M)$ such that $\|\widehat{\sigma}\|_{\rm cb} \le 1$ and $\widehat{\sigma}_{/H_1(R_*)} = \sigma$.

Indeed, we have the following isometric identifications:

$$\begin{aligned} CB\big(H_1(R_*), M\big) &= \big(H_1(R_*) \widehat{\otimes} M_*\big)^* \quad \text{by (1.2)},\\ &= \big(\big(H_1 \widehat{\otimes} R_*\big) \widehat{\otimes} M_*\big)^* \quad \text{by Lemma 4.1(i)},\\ &= \big(H_1 \widehat{\otimes} \big(R \overline{\otimes} M\big)_*\big)^* \quad \text{by [7, Theorem 3.2]},\\ &= \big(H_1\big(\big(R \overline{\otimes} M\big)_*\big)\big)^* \quad \text{by Lemma 4.1(i) again;} \end{aligned}$$

similarly, $CB(L_1(R_*), M) = (L_1((R\overline{\otimes}M)_*))^*$. Since $H_1((R\overline{\otimes}M)_*) \subset L_1((R\overline{\otimes}M)_*)$ isometrically, our claim (4.3) simply follows from Hahn-Banach. We let $y \in H_1(R_*) \otimes B(H)$ and we denote by $\widehat{y} \in L_1(R_*) \otimes B(H)$ its image obtained by canonical embedding. Then, according to (4.3), we have

$$\|\sigma \cdot \pi(y)\| = \|\widehat{\sigma} \cdot \pi(\widehat{y})\| \le \delta(\widehat{y}).$$

Taking the supremum over commuting pairs (σ, π), we obtain $\delta(y) \le \delta(\widehat{y})$. Now observe that since $L_1(R_*)$ is the predual of the von Neumann algebra $L_\infty \overline{\otimes} R$, we have $\delta(\widehat{y}) = \|\widehat{y}\|_{L_1(R_*) \otimes_{\min} B(H)}$ by Theorem 2.1 (see also Remark 2.5). Since

$H_1(R_*) \otimes_{\min} B(H) \subset L_1(R_*) \otimes_{\min} B(H)$ isometrically, we finally obtain $\delta(y) \leq \|y\|_{H_1(R_*)\otimes_{\min} B(H)}$. The converse inequality is obvious, and hence this concludes the proof of (i).

We now prove (ii). We use the property that the operator space $H_1(R_*)$ is locally reflexive. This claim can be justified as follows. First, any subspace of a locally reflexive operator space is locally reflexive itself (see [10] or use Lemma 2.9). Second, $L_1(R_*)$ is locally reflexive because its dual space $L_\infty \overline{\otimes} R$ is a von Neumann algebra (see [6]; see also Theorem 2.10). We let $u\colon Ha(R) \to B(H)$ be a finite rank bounded operator. By Lemma 4.1(ii), we can regard u as an element of $H_1(R_*)^{**} \otimes B(H)$. By Lemma 2.8 and the local reflexivity of $H_1(R_*)$, the norm of u inside $(H_1(R_*) \otimes_{\min} B(H))^{**}$ is $\leq \|u\|_{\text{cb}}$. By Goldstine's lemma, we may thus find a net $u_i\colon Ha(R) \to B(H)$ of finite rank operators such that $\|u_i\|_{\text{cb}} \leq \|u\|_{\text{cb}}$, $u_i \longrightarrow u$ in the w^*-topology of $(H_1(R_*) \otimes_{\min} B(H))^{**}$, and each u_i is represented by an element of $H_1(R_*) \otimes B(H)$ (equivalently, each u_i is w^*-continuous). In particular, $u_i(y) \longrightarrow u(y)$ weakly in $B(H)$ for any $y \in Ha(R)$. By a standard convexity argument, we can actually assume that $\|u_i(y) - u(y)\| \longrightarrow 0$ for any $y \in Ha(R)$. Combining part (i) proved above and Theorem 2.3(ii), we see that $\gamma(u_i) \leq \|u_i\|_{\text{cb}}$, from which we get $\gamma(u_i) \leq \|u\|_{\text{cb}}$. □

Remark 4.4. Let $C = C(\mathbf{T}) \subset L_\infty$ be the space of all continuous functions from $\mathbf{T}$ into $\mathbb{C}$, and let $A = C \cap H_\infty \subset C$ be the so-called disc algebra. Then, taking into account the classical identity $H_1 = (C/A)^*$, it follows from Theorem 4.2(i) that for any finite rank bounded operator $u\colon C/A \to B(H)$, we have $\|u\|_{\text{cb}} = \gamma(u)$.

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