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THE ANALYTIC RANK OF $J_0(q)$ AND ZEROS OF AUTOMORPHIC L-FUNCTIONS

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1. Introduction. This paper is motivated by the conjecture of Birch and Swinnerton-Dyer relating the rank of the Mordell-Weil group of an abelian variety defined over a number field with (in its crudest form) the order of vanishing of its Hasse-Weil L-function at the central critical point. Mestre [Mes] began the study of the implications of this conjecture towards providing upper bounds for the rank. He used "explicit formulae" similar to that of Riemann-Weil and assumed the analytic continuation and (perhaps more significantly) the Riemann hypothesis for those L-functions.

Brumer [Br1] first studied the special case of the Jacobian variety $J_0(q)$ of the modular curve $X_0(q)$. This is an abelian variety defined over $\mathbf{Q}$ of dimension about $q/12$. Here analytic continuation is known, by the work of Eichler and Shimura [Sh1]. Assuming only the Riemann hypothesis for the L-functions of automorphic forms (of weight 2 and level q), Brumer proved

$$\operatorname{rank}_{\mathrm{a}} J_0(q) \leqslant \left(\frac{3}{2}+o(1)\right) \dim J_0(q)$$

and conjectured that

$$\operatorname{rank} J_0(q) = \operatorname{rank}_{\mathrm{a}} J_0(q) \sim \frac{1}{2} \dim J_0(q)$$

(based on the fact that the sign of the functional equation for the automorphic L-functions of weight 2 and level q is approximately half the time $+1$ and half the time -1).

Other authors, notably Murty [Mur] (who first applied the Petersson formula in this context), considered the same problem. Most recently, Luo, Iwaniec, and Sarnak [LIS], using the same assumptions, proved an estimate

$$\operatorname{rank}_{\mathrm{a}} J_0(q) \leqslant \big(c+o(1)\big) \dim J_0(q)$$

for some (explicit) constant $c<1$. This turns out to be quite significant in light of the general conjectures of Katz and Sarnak [KaS] on the distribution of zeros of families of L-functions.

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This paper approaches the same problem with a different emphasis: we wish to avoid all assumptions about the L-functions involved and obtain a bound of the correct order of magnitude. Indeed, we prove the following theorem.

THEOREM 1. *There exists an absolute and effective constant $C > 0$ such that for any prime number q,*

$$\operatorname{rank}_{\mathrm{a}} J_0(q) \leqslant C \dim J_0(q).$$

If the Birch and Swinnerton-Dyer conjecture holds for $J_0(q)$, then

$$\operatorname{rank} J_0(q) \leqslant C \dim J_0(q).$$

This theorem provides the first known unconditional bound for the analytic rank of a family of L-functions that is of the correct order of magnitude, without using the generalized Riemann hypothesis. We were inspired by the unconditional bounds for the analytic rank of twists of elliptic curves obtained by Perelli and Pomykala [PeP].

No remotely comparable upper bound for $\operatorname{rank} J_0(q)$ seems to be accessible by algebraic means today.

The starting point of this work is the factorization of the Hasse-Weil zeta function of $J_0(q)$ due to Eichler and Shimura [Sh1] (completed by Carayol at the bad primes):

$$L\big(J_0(q), s\big) = \prod_{f \in S_2(q)^*} L\left(f, s + \frac{1}{2}\right) \tag{1}$$

where f ranges over the finite set $S_2(q)^*$ of primitive forms (newforms) f of weight 2 and level q, and $L(f,s)$ is the corresponding Hecke L-function normalized so that the critical line is $\operatorname{Re}(s) = 1/2$.

Hence the order of vanishing of the L-function of $J_0(q)$ at $s = 1/2$ is the sum of the order of vanishing of the Hecke L-functions at $s = 1/2$,

$$\operatorname{rank}_{\mathrm{a}} J_0(q) = \sum_{f \in S_2(q)^*} \operatorname{ord}_{s=1/2} L(f,s),$$

and if the Birch and Swinnerton-Dyer conjecture holds, then

$$\operatorname{rank} J_0(q) = \sum_{f \in S_2(q)^*} \operatorname{ord}_{s=1/2} L(f,s).$$

Thus our main theorem is equivalent with the following theorem.

THEOREM 2. *There exists an absolute and effective constant $C > 0$ such that for any prime number q, we have*

$$\sum_{f \in S_2(q)^*} \operatorname{ord}_{s=1/2} L(f,s) \leqslant C \big|S_2(q)^*\big|.$$

The strategy that we use is based on the explicit formula, except that a much tighter control of the possible zeros outside the critical line is required. This is obtained by means of the density Theorem 4 for zeros of automorphic L-functions with imaginary parts as close as $1/(\log q)$, which is the crucial scale in this problem. This density theorem is similar to one proved by Selberg [Sel] for Dirichlet characters and is based on the study of a mollified second moment of values of the L-functions close to the critical line (see below for details).

This proof is carried out in Sections 4 and 5 after some important preliminary results in Section 3. While we were working on this result, it was suggested to us by Iwaniec that the fundamental estimate in the proof of Theorem 1 (for a mollified second moment of values of the L-functions) can also be adapted to prove *nonvanishing* results for the special values $L(f,1/2)$, analogous to part of [IS]. Specifically, this yields the following theorem.

THEOREM 3. *For any $\varepsilon > 0$ and any q prime large enough (in terms of ε), we have*

$$\left|\left\{ f \in S_2(q)^* \,\middle|\, L\left(f, \frac{1}{2}\right) \neq 0 \right\}\right| \geqslant \left(\frac{1}{6} - \varepsilon\right) \left|S_2(q)^*\right|.$$

In [KM3], the case of the special values of the derivatives $L'(f,1/2)$, for forms with $L(f,1/2) = 0$, is treated by the same method. We refer to this paper for some more details.

Duke [Du] proved that the number of forms f with $L(f,1/2) \neq 0$ was at least a positive multiple of $q/(\log q)^2$. Independently, Vanderkam [V] proved that there is a positive proportion (although with smaller constant) of forms with $L(f,1/2) \neq 0$.

This provides a lower bound for the dimension of the winding quotient of Merel [Me]. In particular, the work of Kolyvagin and Logachev [KoLo] implies that there is a quotient of $J_0(q)$ defined over $\mathbf{Q}$ with finite Mordell-Weil group and dimension greater than or equal to $(1/6 + o(1)) \dim J_0(q)$.

The work of Iwaniec and Sarnak [IS] contains, among many other results for various families of L-functions, a proof of the much more difficult fact that $1/6$ can be replaced by $1/4$. Moreover, $1/4$ is the natural barrier in this problem in the sense (the original motivation of [IS]) that any constant greater than $1/4$ (with some additional lower bound on $L(f,1/2)$, which is proved in [IS] to hold for $1/4$) would prove that Landau-Siegel zeros do not exist for Dirichlet L-functions of quadratic characters or, equivalently, provide an effective lower bound for class numbers of imaginary quadratic fields

$$h\big(\mathbf{Q}(\sqrt{-D}\,)\big) \gg \frac{\sqrt{D}}{(\log D)^2}$$

for $D > 0$.

Acknowledgements. The first version of this paper, [KM1] and [KM2], was completed almost two years ago. The current text is based on [K]. Some density theorems

of independent interest that were originally part of [KM2] will be published separately.

More recently, much progress has been made in the direction of upper bounds for $\mathrm{rank}_a\, J_0(q)$. First, a possible value of the constant C can actually be computed. Using the methods of this paper, with some improvements to obtain a better result, it is shown in [KM4] that we can take $C = 6.5$.

Also, a very recent work, in collaboration with J. Vanderkam [KMV1], based on different techniques and more sophisticated arguments, has shown that $C = 1.18191$ is attainable. Observe that this is better than Brumer's original bound using the Riemann hypothesis.

We wish to thank É. Fouvry for numerous discussions and suggestions, and particularly H. Iwaniec for also showing us some of his preliminary notes for [IS].

2. Notation and preliminaries. Throughout this paper, unless otherwise specified, q is a fixed (large) prime.

Let $S_2(q)$ (resp., $S_2(q)^*$) be the space (resp., the finite set) of holomorphic weight-2 cusp forms of level q (resp., primitive weight-2 forms of level q). Recall (from [Sh1, Section 1]) that

$$\dim S_2(q) = \dim J_0(q) = \left|S_2(q)^*\right| \sim \frac{q}{12},$$

the last equality because, as $S_2(1) = 0$, there are no old forms of level q and weight 2 (we use here that q is prime).

We now list notations and facts that will be used extensively in the sequel. Let $f \in S_2(q)^*$ be given. We write $\lambda_f(n)$ for its Hecke eigenvalues, which also give the Fourier expansion of f at infinity:

$$f(z) = \sum_{n\geqslant 1} n^{1/2}\lambda_f(n)e(nz), \quad \text{with } \lambda_f(1) = 1,\ \lambda_f(n) \in \mathbf{R}. \tag{2}$$

Deligne's bound (in the particular case of weight 2, this is due to Eichler-Shimura [Sh1]) for the coefficients of holomorphic cusp forms takes the form

$$\left|\lambda_f(n)\right| \leqslant \tau(n). \tag{3}$$

Recall also that the Hecke L-function of a primitive form f is defined by

$$L(f,s) = \sum_{n\geqslant 1}\lambda_f(n)n^{-s} = \prod_p \left(1-\lambda_f(p)p^{-s}+\varepsilon_q(p)p^{-2s}\right)^{-1}. \tag{4}$$

It satisfies the following functional equation: let

$$\Lambda(f,s) = \left(\frac{\sqrt{q}}{2\pi}\right)^s \Gamma\left(s+\frac{1}{2}\right) L(f,s);$$

then

$$\Lambda(f,s)=\varepsilon_f\Lambda(f,1-s) \tag{5}$$

where ε_f, the sign of the functional equation, is ± 1. For this and other basic facts about Hecke L-functions, see, for instance, [I].

The Euler product representation is equivalent with the following multiplicativity property of the coefficients $\lambda_f(n)$: for any integers $n\geqslant 1$, $m\geqslant 1$,

$$\lambda_f(n)\lambda_f(m)=\sum_{d\mid(n,m)}\varepsilon_q(d)\lambda_f\left(\frac{nm}{d^2}\right). \tag{6}$$

In particular, $n\mapsto\lambda_f(n)$ is multiplicative and $\lambda_f(\delta m)=\lambda_f(\delta)\lambda_f(m)$ if $\delta\mid q$. By Möbius inversion, (6) yields another useful formula,

$$\lambda_f(nm)=\sum_{d\mid(n,m)}\varepsilon_q(d)\mu(d)\lambda_f\left(\frac{n}{d}\right)\lambda_f\left(\frac{m}{d}\right). \tag{7}$$

If p is a prime $\neq q$, we write $1-\lambda_f(p)X+X^2=(1-\alpha_pX)(1-\beta_pX)$, so

$$\lambda_f(p)=\alpha_p+\beta_p. \tag{8}$$

The bound (3) is equivalent (for n coprime with the level) with the assertion that $|\alpha_p|=1$ for all $p\neq q$. For $p=q$, the p-factor of $L(f,s)$ is of degree at most 1, and we let $\alpha_p=\lambda_f(p)$, which is shown to be of modulus at most 1 (actually, smaller), and $\beta_p=0$.

In addition, we require the Dirichlet series expansion for the logarithmic derivative of $L(f,s)$. From the Euler product, using the factorization of the local factors, it follows that

$$-\frac{L'}{L}(f,s)=\sum_{n\geqslant 1}b_f(n)\Lambda(n)n^{-s} \tag{9}$$

with coefficients given by

$$b_f(n)=\begin{cases}0, & \text{if } n \text{ is not a power of a prime,}\\ \alpha_p^m+\beta_p^m, & \text{if } n=p^m.\end{cases} \tag{10}$$

We introduce the notation

$$\omega_f=\frac{1}{4\pi(f,f)} \tag{11}$$

where $(\cdot,\cdot)$ denotes the Petersson inner product on $S_2(q)$,

$$(f,g)=\int_{\Gamma_0(q)\backslash\mathbf{H}}f(z)\overline{g(z)}\,\frac{dx\,dy}{y^2}$$

for any nonzero cusp form $f \in S_2(q)$ (we call this the *harmonic weight*) and define the summation symbol $\sum^h$ by

$$\sum_{f\in S_2(q)^*}^h \alpha_f = \sum_{f\in S_2(q)^*} \omega_f \alpha_f$$

for any family (α_f) of complex numbers. It turns out that (f, f) is of size about $\operatorname{Vol} X_0(q)$, and so about $\dim J_0(q)$; this behaves asymptotically like a probability measure, that is, we have

$$\sum_{f\in S_2(q)^*}^h 1 \sim 1$$

as q tends to infinity. This weight is fundamental to our work because of the following formula due to Petersson (see [I] for instance), which expresses the so-called Δ-symbol,

$$\Delta(m,n) = \sum_{f\in S_2(q)^*}^h \lambda_f(m)\lambda_f(n),$$

in a very convenient way for further analytical manipulations:

$$\Delta(m,n) = \delta(m,n) - 2\pi \sum_{q|c} c^{-1} S(m,n;c) J_1\left(\frac{4\pi\sqrt{mn}}{c}\right)$$

where J_1 is the Bessel function and $S(m,n;c)$ is a classical Kloosterman sum. Using the estimates

$$\left|S(m,n;c)\right| \leqslant \tau(c)(m,n,c)^{1/2}\sqrt{c}, \qquad J_1(x) \ll x$$

(the first being Weil's bound for Kloosterman sums [Wei], the second being elementary), we derive

$$\Delta(m,n) = \delta(m,n) + O\Big((m,n,q)\big(\log(m,n)\big)^2 (mn)^{1/2} q^{-3/2}\Big). \tag{12}$$

We do not need any better bounds in this work (compare [KM3] and [IS] for cases where a more precise analysis with the Kloosterman sums is needed).

3. From harmonic average to natural average, I

3.1. Averages. We deal quite extensively with sums over $f \in S_2(q)^*$. The following notation is designed to emphasize the underlying structure. We usually assume that a family $\alpha = (\alpha_f)$ of complex numbers is given, where f runs over all $f \in S_2(q)^*$ for all levels q (or sometimes restricted to square-free or prime levels). We then introduce the *natural* averaging operator

$$A[\alpha] = \sum_{f\in S_2(q)^*} \alpha_f$$

where we only sum over forms of a fixed level and consider the behavior of $A[\alpha]$ as a function of the level q asymptotically as q gets large.

Similarly, we define the *harmonic* averaging operator

$$A^h[\alpha] = \sum_{f \in S_2(q)^*}^h \alpha_f.$$

Suppose we have a family $\alpha = (\alpha_f)$ of complex numbers for all $f \in S_2(q)^*$ with prime level q, and we know the behavior of the weighted sum

$$A^h[\alpha] = \sum_{f \in S_2(q)^*}^h \alpha_f$$

(for instance, we have an asymptotic formula for q going to infinity) but wish to obtain the same information for the natural sum

$$A[\alpha] = \sum_{f \in S_2(q)^*} \alpha_f.$$

Since, by Petersson's formula (12) for $m = n = 1$, $A^h[1] = 1 + O(q^{-3/2})$, we expect that when α is well distributed and not biased against the Petersson inner product, we should have

$$A[\alpha] \sim \dim J_0(q) A^h[\alpha],$$

meaning that ω_f and α_f act here as independent random variables would.

In this section, we build a method (suggested to us by Iwaniec) to approach this problem and prove a result that solves part of the problem for quite general vectors α. This reduces to another estimate, which has to be supplied independently in each case.

3.2. The symmetric square. The harmonic weight ω_f, which is required to express the Δ-symbol of the modular forms by Petersson's formula, is related to the special value of the symmetric square L-function at $s = 1$, which is the edge of the critical strip (in our "analytic" normalization). This is essentially due to Shimura [Sh2].

The symmetric square L-function of f is the Dirichlet series $L(\mathrm{Sym}^2 f, s)$ defined by

$$L\big(\mathrm{Sym}^2 f, s\big) = \zeta_q(2s) \sum_{n \geqslant 1} \lambda_f\big(n^2\big) n^{-s} \tag{13}$$

where ζ_q is the Riemann zeta function with the Euler factor at q removed. We write $\rho_f(n)$ for the coefficients of this Dirichlet series. The relation we seek is given by the following formula:

$$4\pi(f,f) = \frac{\dim J_0(q)}{\zeta(2)} L\big(\mathrm{Sym}^2 f, 1\big) + O\big((\log q)^3\big) \tag{14}$$

(uniformly in f as the prime q tends to infinity; for a proof see [K]).

The following lemma summarizes some properties of the coefficients $\rho_f(n)$.

Lemma 1. *For any $n \geqslant 1$, we have*

$$\rho_f(n) = \sum_{\ell m^2 = n} \varepsilon_q(m)\lambda_f(\ell^2), \tag{15}$$

$$\lambda_f(n^2) = \sum_{\ell m^2 = n} \mu(m)\varepsilon_q(m)\rho_f(\ell), \tag{16}$$

and, in particular, $\rho_f(n) = \lambda_f(n^2)$ for n square-free. Moreover, $L(\mathrm{Sym}^2 f, s)$ has an Euler product expansion of degree 3,

$$L(\mathrm{Sym}^2 f, s) = \prod_{(p,q)=1} (1-\alpha_p^2 p^{-s})^{-1}(1-p^{-s})^{-1}(1-\alpha_p^{-2} p^{-s})^{-1} \prod_{p \mid q}(1-\alpha_p^2 p^{-s})^{-1}$$

where α_p is as in (8). Finally, for all $n \geqslant 1$, we have

$$|\rho_f(n)| \leqslant \tau(n)^2. \tag{17}$$

The last estimate is proved using Deligne's bound $|\lambda_f(n)| \leqslant \tau(n)$ and the Euler product.

Lemma 2. *For all q prime and all $f \in S_2(q)^*$, we have*

$$L(\mathrm{Sym}^2 f, 1) \ll (\log q)^3, \tag{18}$$

$$L(\mathrm{Sym}^2 f, 1) \gg (\log q)^{-1}. \tag{19}$$

The implied constants are absolute in both cases.

The (deeper) lower bound is the main result of [GHL]. The fact that q is prime ensures that f is not a monomial form, as such a form (over $\mathbf{Q}$) cannot have square-free conductor. The upper bound is much easier and well known (see [K, Lemma 2], for instance). In particular, we have uniformly for $f \in S_2(q)^*$,

$$\omega_f \ll \frac{\log q}{q}. \tag{20}$$

We require a property of *almost orthogonality* of the coefficients of the symmetric square L-functions of the forms $f \in S_2(q)^*$. It is implicitly contained in the second part of [DK], where it was developed for other applications.

Proposition 1. *Let $q \geqslant 1$ be any square-free integer, and let $N \geqslant q^9$ be a real number. The inequality*

$$\sum_{f \in S_2(q)^*} \Big| \sum_{n \leqslant N} a_n \rho_f(n) \Big|^2 \ll N(\log N)^{15} \sum_{n \leqslant N} |a_n|^2 \tag{21}$$

holds for any finite family $(a_n)_{1 \leqslant n \leqslant N}$ of complex numbers, with an absolute implied constant.

We deduce the following corollary.

COROLLARY 1. *Let $N \geqslant q^9$ be a real number and let $(a(n))_{n\sim N}$ be any complex numbers that satisfy*

$$a(n) \ll \frac{\big(\tau(n)\log n\big)^A}{n}$$

for some constant $A > 0$. There exists a constant $D = D(A) \geqslant 0$ such that

$$\sum_{f\in S_2(q)^*}\Big|\sum_{n\sim N} a(n)\lambda_f\big(n^2\big)\Big|^2 \ll (\log N)^D$$

(with an absolute implied constant).

Proof. The point is that the assumption on the a_n means that we are essentially on the line $\mathrm{Re}(s) = 1$ (or beyond), and in this region the symmetric square behaves as the series

$$\sum_{n\geqslant 1}\lambda_f\big(n^2\big)n^{-s}.$$

In exacting details, we have from (15) that

$$\begin{aligned}\sum_{f\in S_2(q)^*}\Big|\sum_{n\sim N} a(n)\lambda_f\big(n^2\big)\Big|^2 &= \sum_{f\in S_2(q)^*}\Big|\sum_{n\sim N}\sum_{\ell m^2=n}\mu(m)\varepsilon_q(m)\rho_f(\ell)a(n)\Big|^2\\ &= \sum_{f\in S_2(q)^*}\Big|\sum_{\ell\leqslant 2N}\rho_f(\ell)\tilde a(\ell)\Big|^2\end{aligned}$$

where

$$\tilde a(\ell) = \sum_{\sqrt{N/\ell}<m\leqslant\sqrt{2N/\ell}}\mu(m)\varepsilon_q(m)a\big(\ell m^2\big).$$

Now we derive from the assumption a bound

$$\tilde a(\ell) \ll (N\ell)^{-1/2}(\log\ell)^D$$

(for some $D \geqslant 0$, with an absolute implied constant) and, hence, the result on applying the mean-value estimate of Proposition 1 to the coefficients $\tilde a(\ell)$. □

3.3. Removing the harmonic weight. We assume that $\alpha = (\alpha_f)$ satisfies the conditions

$$A^h\big[|\alpha_f|\big] \ll (\log q)^A \quad \text{(for some absolute } A > 0), \tag{22}$$

$$\operatorname*{Max}_{f\in S_2(q)^*}\big|\omega_f\alpha_f\big| \ll q^{-\delta} \quad \text{(for some } \delta > 0), \tag{23}$$

as the level q (prime) tends to infinity.

Remark. Neither condition is very restrictive in practice. The first one is interpreted as saying that $|\alpha_f|$ is "almost" bounded, and this can often be achieved by some normalization. If this is true, the second condition is fairly reasonable, since we have shown in (20) that $\omega_f \ll (\log q)q^{-1}$. In other words, by normalizing if necessary, both conditions can be expected to hold whenever the size of α_f does not increase or oscillate wildly.

We write the unweighted average as a weighted one and replace the Petersson inner product by the special value of the symmetric square (14):

$$\begin{aligned} A[\alpha] &= \sum_{f \in S_2(q)^*}^h 4\pi(f,f)\alpha_f \\ &= \frac{\dim J_0(q)}{\zeta(2)} \sum_{f \in S_2(q)^*}^h L\big(\mathrm{Sym}^2 f, 1\big)\alpha_f + O\big((\log q)^3 A^h\big[|\alpha_f|\big]\big). \end{aligned} \tag{24}$$

We wish to replace the value of the symmetric square by a partial sum of the Dirichlet series. This can be done by a long enough sum, say, of length y. Then the sum above is essentially a finite sum of averages over the α_f, twisted by symmetric square coefficients $\rho_f(n)$:

$$\sum_{n \leqslant y} \frac{1}{n} A^h\big[\rho_f(n)\alpha_f\big].$$

If, by any chance, the methods that give us control over the average $A^h[\alpha_f]$ (corresponding to $n = 1$) also apply to the twisted ones *in the range* $n < y$, then we are done. Unfortunately, in applications this is only the case for very small values of y, say, $y = q^\delta$ for very small $\delta > 0$. On the Riemann hypothesis (more precisely, the Lindelöf hypothesis suffices for this purpose), we can recover the L-function from such a short sum; but individually we can only do this with y much larger ($y = q^2$ or maybe $y = q$) and indeed too large for our applications.

However, we can exploit the average over f involved by using the mean value estimate of Proposition 1. The fact that this requires also a long sum in n is not a problem here because we are looking at the symmetric square at a point on the edge of the critical strip where the Dirichlet series almost converges absolutely. Then the extra length needed to enter the effective range of n for the mean-value estimate does not matter, much as the partial sums

$$\sum_{n < q^\delta} n^{-1}$$

of the harmonic series are of the same size as q tends to infinity for any fixed $\delta > 0$.

Now we implement this idea. Let $\alpha = (\alpha_f)_{f \in S_2(q)^*}$ be given for all q prime, satisfying the conditions (22) and (23). Since the conductor of $\mathrm{Sym}^2 f$ for $f \in S_2(q)^*$

is q^2, the functional equation and the usual estimates give the approximation

$$L\big(\mathrm{Sym}^2 f, 1\big) = \omega_f(y) + O\big(q^2 y^{-1}\big) \tag{25}$$

(with an absolute implied constant), where

$$\omega_f(y) = \sum_{n \leqslant y} \rho_f(n) n^{-1}.$$

We assume $\log y = O(\log q)$, say, $y < q^{10}$.

Now let $x < y$ be given. The partial sum is further decomposed as

$$\omega_f(y) = \omega_f(x) + \omega_f(x, y)$$

where

$$\omega_f(x, y) = \sum_{x < n \leqslant y} \rho_f(n) n^{-1}.$$

We consider here the weighted average built with the tail, namely,

$$A^h\big[\omega_f(x, y)\alpha_f\big] = \sum_{f \in S_2(q)^*}{}^{h} \omega_f(x, y)\alpha_f.$$

We use Hölder's inequality to separate $\omega_f(x, y)$ and α_f. The former is handled by the following lemma.

LEMMA 3. *Let $r \geqslant 1$ be an integer such that $x^r \geqslant q^{11}$. There exists a positive constant $C = C(r) > 0$ such that*

$$A\big[\omega_f(x, y)^{2r}\big] \ll (\log q)^C$$

where the implied constant is absolute.

The proof starts with some other lemmas. We say that an integer n is square-full if for any prime p dividing n, p^2 divides n; in other words, for all p dividing n, the valuation of p in n is at least 2. Notice that

$$\sum_{n \text{ square-full}} n^{-s} = \prod_p \big(1 + p^{-2s} + p^{-3s} + \cdots\big),$$

which converges absolutely for $\mathrm{Re}(s) > 1/2$, and hence we have

$$\sum_{\substack{n \text{ square-full} \\ n > z}} n^{-1} \ll z^{-1/2} \tag{26}$$

with an absolute implied constant.

LEMMA 4. *For any integer $r \geqslant 1$ and any $f \in S_2(q)^*$, we can write*

$$\omega_f(x, y)^r = \sum_{x^r < mn \leqslant y^r} \lambda_f\big(m^2\big) \frac{c(m, n)}{mn} \tag{27}$$

with $c(m, n) = 0$ unless n can be written

$$n = dn_1, \quad \text{with } d \mid m,\ n_1 \text{ square-full,} \tag{28}$$

and there exists $\gamma = \gamma(r) > 0$ such that

$$|c(m, n)| \leqslant \tau(mn)^\gamma.$$

Moreover, the coefficients c depend on r, x, and y, but not on the form f.

Proof. We proceed by induction on r. For $r = 1$, by (15) we write

$$\begin{aligned}\omega_f(x, y) &= \sum_{x < n \leqslant y} \frac{1}{n} \sum_{\ell m^2 = n} \varepsilon_q(m) \lambda_f\big(\ell^2\big) \\ &= \sum_{x < \ell m^2 \leqslant y} \lambda_f\big(\ell^2\big) \frac{\varepsilon_q(m)}{\ell m^2},\end{aligned}$$

so we can take $c(\ell, m) = 0$ unless m is square and $c(\ell, m^2) = \varepsilon_q(m)$.

Assume that (27) holds for some r and s as claimed, with coefficients c (for r) and c' (for s). Then

$$\begin{aligned}\omega_f(x, y)^{r+s} &= \sum_{\substack{x^r < m_1 n_1 \leqslant y^r \\ x^s < m_2 n_2 \leqslant y^s}} \lambda_f\big(m_1^2\big) \lambda_f\big(m_2^2\big) \frac{c(m_1, n_1) c'(m_2, n_2)}{m_1 n_1 m_2 n_2} \\ &= \sum_{\substack{x^r < m_1 n_1 \leqslant y^r \\ x^s < m_2 n_2 \leqslant y^s}} \sum_{\substack{d \mid m_1^2 \\ d \mid m_2^2}} \lambda_f\left(\frac{m_1^2 m_2^2}{d^2}\right) \frac{\varepsilon_q(d) c(m_1, n_1) c'(m_2, n_2)}{m_1 n_1 m_2 n_2}\end{aligned}$$

by multiplicativity for λ_f.

Now d can be written uniquely as $d = d_1 d_2^2$ with d_1 square-free. Then we have $d \mid m^2$ if and only if $d_1 d_2 \mid m$. Therefore, we can write

$$\begin{cases} m_1 = d_1 d_2 m_1', \\ m_2 = d_1 d_2 m_2', \end{cases}$$

and then

$$\omega_f(x, y)^{r+s} = \sum_{\substack{x^r < d_1 d_2 m_1' n_1 \leqslant y^r \\ x^s < d_1 d_2 m_2' n_2 \leqslant y^s}} \lambda_f\Big(\big(d_1 m_1' m_2'\big)^2\Big) \frac{\varepsilon_q(d_1 d_2) c(d_1 d_2 m_1', n_1) c'(d_1 d_2 m_2', n_2)}{(d_1 d_2)^2 m_1' m_2' n_1 n_2}.$$

Now write $m_0 = d_1 m_1' m_2'$, $n_0 = d_1 d_2^2 n_1 n_2$. By the induction hypothesis, we see that if $c(m_1, n_1) \neq 0$ and $c'(m_2, n_2) \neq 0$, then n_0 can be written as $\delta n_0'$ with $\delta \mid m_0$ and n_0' square-full. (This is not absolutely obvious, because $m_1 m_2$ does not divide m_0, but the extra prime divisors can be pushed to the square-full part.)

Estimating rather trivially the multiplicity of representation of m_0, we find the desired representation. This immediately concludes the induction. □

LEMMA 5. *Let $z \geqslant 1$ be given and let the coefficients $c(m,n)$ be as in Lemma 4 for r. Then there exists $A = A(r) > 0$ such that*

$$\sum_{\substack{x^r < mn \leqslant y^r \\ n > z}} \lambda_f\big(m^2\big) \frac{c(m,n)}{mn} = O\big(z^{-1/2} (\log qz)^A\big).$$

Proof. By Deligne's bound, we have

$$\sum_{\substack{x^r < mn \leqslant y^r \\ n > z}} \lambda_f\big(m^2\big) \frac{c(m,n)}{mn} \leqslant \sum_{x^r < m \leqslant y^r} \frac{\tau(m)}{m} \sum_{\substack{x^r m^{-1} < n \leqslant y^r m^{-1} \\ n > z}} \frac{\big|c(m,n)\big|}{n},$$

but using the condition on the support of $c(m,n)$, the inner sum is

$$\begin{aligned} \sum_{\substack{x^r m^{-1} < n \leqslant y^r m^{-1} \\ n > z}} \frac{\big|c(m,n)\big|}{n} &\leqslant \tau(m)^\gamma \sum_{d \mid m} \frac{1}{d} \sum_{\substack{n \text{ square-full} \\ dn > z}} \frac{\tau(n)^\gamma}{n} \\ &\ll \tau(m)^{\gamma+1} z^{-1/2} (\log z)^A \end{aligned}$$

(by (26)), and the result follows. □

LEMMA 6. *There exist a real number M such that $x^r z^{-1} < M \leqslant y^r z$ and real numbers $c(m)$ such that*

$$\sum_{f \in S_2(q)^*} \omega_f(x,y)^{2r} \ll (\log qz)^B \sum_{f \in S_2(q)^*} \Bigg| \sum_{m \sim M} \lambda_f\big(m^2\big) \frac{c(m)}{m} \Bigg|^2 + O\big(qz^{-1/2} (\log qz)^B\big)$$

and

$$|c(m)| \leqslant \tau(m)^C (\log qm)^C$$

for some $C > 0$.

Proof. By Lemmas 4 and 5,

$$\omega_f(x,y)^{2r} = \sum_{n \leqslant z} \Bigg| \sum_{x^r < mn \leqslant y^r} \lambda_f\big(m^2\big) \frac{c(m,n)}{mn} \Bigg| + O\big(qz^{-1/2} (\log qz)^A\big).$$

Write $\xi_n = \operatorname{sign}\bigl(\sum_{x^r<mn\leqslant y^r}\lambda_f(m^2)c(m,n)/mn\bigr)$ and split the summation over dyadic intervals in m. By Cauchy's inequality and summing over f, the result follows for some M with

$$c(m)=\sum_{x^r m^{-1}<n\leqslant z}\xi_n\frac{c(m,n)}{n}\ll\tau(m)^C(\log qm)^C$$

for some $C>0$, as desired. □

This now easily implies Lemma 3: take $z=q^2$; then the assumption $x^r\geqslant q^{11}$ implies that $M\geqslant q^9$, and we may appeal to the mean value estimate of Corollary 1 to bound the first term with $\log M\ll\log q$.

PROPOSITION 2. *Let (α_f) be complex numbers satisfying conditions (22) and (23), and let $x=q^\kappa$ for some $\kappa>0$. There exists an absolute constant $\gamma=\gamma(\kappa,\delta)>0$ (δ being the exponent in (22)) such that*

$$A^h\bigl[\omega_f(x,y)\alpha_f\bigr]\ll q^{-\gamma}$$

and

$$A\bigl[\alpha_f\bigr]=\frac{\dim J_0(q)}{\zeta(2)}A^h\bigl[\omega_f(x)\alpha_f\bigr]+O\bigl(q^{1-\gamma}\bigr).$$

Proof. Let $r\geqslant 1$ be any integer. By Hölder's inequality, we have (with s the complementary exponent to $2r$, $(2r)^{-1}+s^{-1}=1$)

$$\begin{aligned}A^h\bigl[\omega_f(x,y)\alpha_f\bigr]&=\sum_{f\in S_2(q)^*}{}^{h}\omega_f(x,y)\alpha_f=\sum_{f\in S_2(q)^*}\omega_f\omega_f(x,y)\alpha_f\\&\leqslant A\bigl[\omega_f(x,y)^{2r}\bigr]^{1/2r}\Biggl(\sum_{f\in S_2(q)^*}\bigl(\omega_f|\alpha_f|\bigr)^s\Biggr)^{1/s}\\&\leqslant A^{1/2r}A\bigl[\omega_f(x,y)^{2r}\bigr]^{1/2r}A^h\bigl[|\alpha_f|\bigr]^{1/s}\end{aligned}$$

where we denote

$$A=\operatorname*{Max}_{f\in S_2(q)^*}\omega_f|\alpha_f|.$$

Now take r large enough so that $x^r\geqslant q^{11}$. Then Lemma 3 gives

$$A\bigl[\omega_f(x,y)^{2r}\bigr]^{1/2r}\ll(\log q)^D$$

for some $D=D(\kappa)>0$, while we have, from (23) and (22), respectively,

$$A^{1/2r}\ll q^{-\gamma_0}\quad\text{for some }\gamma_0=\gamma_0(\kappa,\delta)>0,$$

$$A^h\bigl[|\alpha_f|\bigr]\ll(\log q)^C\quad\text{for some absolute constant }C>0.$$

Hence we prove the proposition, the last equality being an immediate corollary of the formula

$$A\big[\alpha_f\big] = \frac{\dim J_0(q)}{\zeta(2)} A^h\big[L\big(\mathrm{Sym}^2 f, 1\big)\alpha_f\big] + O\big((\log q)^3 A^h\big[|\alpha_f|\big]\big)$$

and the decomposition

$$L\big(\mathrm{Sym}^2 f, 1\big) = \omega_f(x) + \omega_f(x, y) + O\big(q^2 y^{-1}\big)$$

applied with $y = q^3$. □

4. Upper bound for the analytic rank of $J_0(q)$. In this section, we prove Theorem 1 via the equivalent form of Theorem 2.

4.1. Reduction to the density theorem. The explicit formulae, discovered in essence by Riemann and later extended and formalized by Weil, were used by Mestre in studying abelian varieties. We use the following variant (compare to [Br1], [PeP], [K]).

PROPOSITION 3. *Let $\psi : \mathbf{R} \longrightarrow \mathbf{R}$ be a C^∞ even function with compact support and let $\hat{\psi}$, $\hat{\psi}(s) = \int_{\mathbf{R}} \psi(x) e^{sx}\, dx$ be its Laplace transform, which is an entire function. Then for any primitive form $f \in S_2(q)^*$, we have*

$$\sum_{\rho} \hat{\psi}\left(\rho - \frac{1}{2}\right) = \psi(0)\log q - 2\sum_{n\geqslant 1} \frac{b_f(n)}{\sqrt{n}}\, \Lambda(n)\psi(\log n) + \frac{1}{2i\pi}\int_{1/2} 2\left(\frac{\Gamma'}{\Gamma}\left(s + \frac{1}{2}\right) - \log 2\pi\right)\hat{\psi}\left(s - \frac{1}{2}\right) ds, \tag{29}$$

the summation on the left-hand side being extended over all zeros ρ of $L(f, s)$ in the critical strip (those with $0 \leqslant \mathrm{Re}(s) \leqslant 1$) counted with multiplicity. The coefficients $b_f(n)$ are defined in (10).

In this chapter, ρ always designates such a nontrivial zero of $L(f, s)$, and we always write

$$\rho = \beta + i\gamma$$

so that $\gamma = \mathrm{Im}(\rho)$, $\beta = \mathrm{Re}(\rho)$. For any α with $0 \leqslant \alpha \leqslant 1$ and any real numbers $t_1 \leqslant t_2$, we define $N(f; \alpha, t_1, t_2)$ to be the number of zeros $\rho = \beta + i\gamma$ of $L(f, s)$, counted with multiplicity, such that

$$\beta \geqslant \alpha, \qquad t_1 \leqslant \gamma \leqslant t_2,$$

and for any $t > 0$, we let

$$N(f; \alpha, t) = N(f; \alpha, -t, t), \qquad N(f, t) = N(f; 0, t).$$

The standard theory of L-functions gives the asymptotic formula

$$N(f,t)\sim\frac{t}{2\pi}\left(\log\frac{qt}{2\pi e}\right)+O\left(\log qt\right).$$

In particular, we are led to expect that the number of zeros (with multiplicity) of $L(f,s)$ within distance $\sim(\log q)^{-1}$ of $1/2$ should be absolutely bounded, at least on average over f. Hence, if we (over) count all those zeros, not only $\rho=1/2$, Theorem 2 should still remain valid.

We fix a test function ψ, which is assumed to be even, nonnegative, compactly supported in $[-1,1]$ with $\psi(0)=1$, and such that $\hat{\psi}(s)$ satisfies the positivity condition

$$\mathrm{Re}\left(\hat{\psi}(s)\right)\geqslant 0 \tag{30}$$

for all $s\in\mathbf{C}$ with $|\mathrm{Re}(s)|\leqslant 1$.

For $\lambda>0$, let $\psi_\lambda(x)=\psi(x/\lambda)$ so that

$$\hat{\psi}_\lambda(s)=\lambda\hat{\psi}(\lambda s).$$

We take $\lambda=\theta\log q$ with $\theta>0$ a fixed parameter (small enough) to be determined later. This parameter λ is used to effect a localization in detecting the zeros around $1/2$ in the explicit formula.

The crucial assumption on ψ is of course (30). Such test functions were constructed by Poitou and others for the purpose of obtaining lower bounds for the discriminant of number fields [Poi].

Remark. In [KM1], a specific test function F, which had been constructed previously by Perelli and Pomykala [PeP], is used. However, in our situation it is not actually necessary to use it (this was observed by Pomykala).

By integration by parts, one infers that for any integer $k\geqslant 1$,

$$\hat{\psi}(s)\ll_k\frac{1}{\left(1+|\mathrm{Im}(s)|\right)^k}\,e^{\mathrm{Re}(s)} \tag{31}$$

where the implied constant depends only on k (and on the specific choice of ψ).

Let q be prime and let $f\in S_2(q)^*$ be a primitive form of level q. Applying the explicit formula (29) to f with the test function ψ_λ, we obtain

$$\sum_\rho\hat{\psi}_\lambda\left(\rho-\frac{1}{2}\right)=\log q-2\sum_{n\geqslant 1}\frac{b_f(n)}{\sqrt{n}}\,\Lambda(n)\psi_\lambda(\log n)+O(1).$$

We estimate the integral in (29) by

$$\frac{1}{2i\pi}\int_{(1/2)}2\left(\frac{\Gamma'}{\Gamma}\left(s+\frac{1}{2}\right)-\log 2\pi\right)\hat{\psi}_\lambda\left(s-\frac{1}{2}\right)ds\ll\lambda\int_{-\infty}^{+\infty}\left(1+|u|\right)\hat{\psi}(\lambda iu)\,du\ll 1$$

uniformly in λ (we use $\Gamma'/\Gamma(s) \ll \log|s|$ for $\mathrm{Re}(s) = 1$ together with (31)).

Then we isolate the multiplicity of the zero at $1/2$. We further distinguish among the remaining zeros ρ between those that are close to $1/2$—precisely, those with $|\beta - 1/2| \leqslant \lambda^{-1}$—and the others. On the right-hand side, we use the fact that Λ is supported on powers of primes, and we put the primes apart from the squares and higher powers. This way we rewrite the outcome of the explicit formula:

$$\lambda\hat{\psi}(0)\,\mathrm{ord}_{s=1/2}\,L(f,s) + \Xi_1(f,\lambda) + \Xi_2(f,\lambda) = \log q - 2S_1(f,\lambda) - 2S_2(f,\lambda) + O(1) \tag{32}$$

with

$$\begin{aligned} \Xi_1(f,\lambda) &= \lambda \sum_{|\beta-1/2|\leqslant\lambda^{-1}} \hat{\psi}\left(\lambda\left(\rho - \frac{1}{2}\right)\right), \\ \Xi_2(f,\lambda) &= \lambda \sum_{|\beta-1/2|>\lambda^{-1}} \hat{\psi}\left(\lambda\left(\rho - \frac{1}{2}\right)\right), \end{aligned} \tag{33}$$

$$\begin{aligned} S_1(f,\lambda) &= \sum_p \frac{\lambda_f(p)}{\sqrt{p}}\,(\log p)\psi_\lambda(\log p), \\ S_2(f,\lambda) &= \sum_{n\geqslant 2}\sum_p \frac{\lambda_f(p^n)}{p^{n/2}}\,(\log p)\psi_\lambda\big(\log p^n\big). \end{aligned} \tag{34}$$

Each term is treated separately. First, since ψ has compact support in $[-1,1]$ and $|b_f(n)| \leqslant 2$ for all n, we have

$$S_2(f,\lambda) \ll \sum_{p\leqslant\exp(\lambda/2)} \frac{\log p}{p} + \sum_{3\leqslant n\leqslant\lambda}\ \sum_{\log p\leqslant\lambda/n} \frac{\log p}{p^{n/2}} \ll \lambda.$$

Now, in (32), we take the real part. For a zero ρ appearing in $\Xi_1(f,\lambda)$, we have $|\mathrm{Re}\,\lambda(\rho - 1/2)| \leqslant 1$, and hence

$$\Xi_1(f,\lambda) \geqslant 0$$

by the positivity property (30) of the test function ψ. Therefore, we can drop this term by positivity and get

$$\lambda\hat{\psi}(0)\,\mathrm{ord}_{s=1/2}\,L(f,s) \leqslant \log q - 2S_1(f,\lambda) + \mathrm{Re}\big(\Xi_2(f,\lambda)\big) + O(\lambda).$$

Intuitively, this application of positivity should not affect the chances of proving the result being sought, since the number of zeros dropped in the sum $\Xi_1(f,\lambda)$, on average over f, should be bounded.

Performing the average over f, we consequently have

$$\lambda\hat{\psi}(0)\sum_{f\in S_2(q)^*}\mathrm{ord}_{s=1/2}L(f,s)\leqslant(\log q)\dim J_0(q)-2\sum_{f\in S_2(q)^*}S_1(f,\lambda)+\sum_{f\in S_2(q)^*}\mathrm{Re}\big(\Xi_2(f,\lambda)\big)+O(\lambda q).\tag{35}$$

LEMMA 7. *Assume $\theta<3/4$. There exists a constant $\delta=\delta(\theta)>0$ such that*

$$\sum_{f\in S_2(q)^*}S_1(f,\lambda)\ll q^{1-\delta}.\tag{36}$$

Proof. We write

$$\sum_{f\in S_2(q)^*}S_1(f,\lambda)=A\big[S_1(f,\lambda)\big]$$

and proceed to estimate this by the method of Section 3 (as an illustration of this technique). We might also quote [Br1], where the trace formula is used to prove the corresponding result.

We need to check conditions (22) and (23) to apply Proposition 2. The individual bound (23) is easy. We have

$$S_1(f,\lambda)=\sum_p\frac{\lambda_f(p)}{\sqrt{p}}\psi_\lambda(\log p).$$

The support of ψ limits the summation to primes with $\log p\leqslant\lambda$, that is, $p\leqslant q^\theta$, so

$$S_1(f,\lambda)\ll\sum_{p\leqslant q^\theta}p^{-1/2}\ll q^{\theta/2}.$$

On the other hand, $\omega_f\ll(\log q)q^{-1}$ by (20), and (23) follows.

Next we estimate the harmonic average $A^h[|S_1(f,\lambda)|]$. Since the sum $S_1(f,\lambda)$ is real, Cauchy's inequality implies

$$A^h\big[|S_1(f,\lambda)|\big]\leqslant A^h\big[S_1(f,\lambda)^2\big]^{1/2}A^h[1]^{1/2}\ll A^h\big[S_1(f,\lambda)^2\big]^{1/2}.$$

Now we compute

$$\begin{aligned}S_1(f,\lambda)^2&=\sum_{p,p'}\frac{\lambda_f(p)\lambda_f(p')}{\sqrt{pp'}}(\log p)(\log p')\psi_\lambda(\log p)\psi_\lambda(\log p')\\&=\sum_p\frac{\lambda_f(p)^2}{p}(\log p)^2\psi_\lambda(\log p)^2\\&\quad+\sum_{p\neq p'}\frac{\lambda_f(pp')}{\sqrt{pp'}}(\log p)(\log p')\psi_\lambda(\log p)\psi_\lambda(\log p').\end{aligned}$$

Now we use $\lambda_f(p)^2 = \lambda_f(p^2)+1$, which is true for all primes p occurring in the summation since $\psi_\lambda(\log p) = 0$ for $\log q > \lambda$, that is, for $p > q^\theta$, and we get

$$A^h\big[S_1(f,\lambda)^2\big] = \sum_p \frac{(\log p)^2}{p}\,\psi_\lambda(\log p)^2\big(\Delta(1,1)+\Delta\big(1,p^2\big)\big)$$
$$+\sum_{p\neq p'} \frac{(\log p)(\log p')}{\sqrt{pp'}}\,\psi_\lambda(\log p)\psi_\lambda(\log p')\Delta(1,pp').$$

Since evidently $pp' \neq 1$, we obtain from (12) that

$$A^h\big[S_1(f,\lambda)^2\big] \ll \sum_{p\leqslant q^\theta} \frac{(\log p)^2}{p} + \frac{(\log q)^2}{q^{3/2}}\Bigg|\sum_{p\leqslant q^\theta}(\log p)\Bigg|^2 \ll (\log q)^2$$

since $2\theta < 3/2$.

From Proposition 2, we conclude that there exists a constant $\delta = \delta(\theta)$ such that

$$A\big[S_1(f,\lambda)\big] = \frac{\dim J_0(q)}{\zeta(2)}A^h\big[\omega_f(x)S_1(f,\lambda)\big] + O\big(q^{1-\delta}\big)$$

with $x = q^\kappa$, $\kappa < 1/4$ being a (small) parameter to be chosen below.

From the definition of $\omega_f(x)$, we derive

$$A^h\big[\omega_f(x)S_1(f,\lambda)\big] = \sum_{d\ell^2\leqslant x}\frac{\lambda_f\big(d^2\big)}{d\ell^2}\sum_p \frac{\lambda_f(p)}{\sqrt{p}}(\log p)\psi_\lambda(\log p)$$
$$= \sum_{d\ell^2\leqslant x}\frac{1}{d\ell^2}\sum_p\frac{(\log p)}{\sqrt{p}}\,\psi_\lambda(\log p)\Delta\big(p,d^2\big)$$
$$\ll \frac{(\log q)^2}{q^{3/2}}\sum_{d\ell^2\leqslant x}\frac{1}{\ell^2}\sum_{p\leqslant q^\theta}(\log p)$$
$$\ll (\log q)^2 q^{\theta+\kappa-3/2},$$

and the lemma follows by taking κ small enough that the exponent is negative. □

Thus it only remains to estimate the contribution of Ξ_2, the sum over zeros not too close to $1/2$. Of course, on the generalized Riemann hypothesis, those do not exist. Then, taking the upper bound above (35), we immediately obtain a weak form of Brumer's result, namely,

$$\sum_{f\in S_2(q)^*} \operatorname{ord}_{s=1/2} L(f,s) \ll \dim J_0(q)$$

for q prime. Up to this point, the treatment is basically the same as Brumer's. But handling Ξ_2 without appealing to the Riemann hypothesis is precisely the crux of the matter. It is possible to show that if there are zeros in the region $|\beta - 1/2| > \lambda^{-1}$, then they are very few in number in a very precise sense, which we now describe.

THEOREM 4. *Let q be a prime number. There exists an absolute constant $A > 0$ such that for any real numbers t_1 and t_2 with*

$$t_1 < t_2,$$
$$t_2 - t_1 \geqslant \frac{1}{\log q},$$

for any $\alpha \geqslant 1/2 + (\log q)^{-1}$, and for any c, $0 < c < 1/4$, one has

$$\sum_{f \in S_2(q)^*} N(f; \alpha, t_1, t_2) \ll \big(1 + |t_1| + |t_2|\big)^A q^{1-c(\alpha-1/2)} (\log q)(t_2 - t_1). \tag{37}$$

The implied constant depends only on c.

The bulk of this section is devoted to proving this result.

Remark. In this density theorem, only the q-aspect is taken into consideration, and this statement is indeed trivial with respect to t_1 and t_2. However, it is important (as the deduction of the upper bound from the density theorem shows) that the bounds obtained be at most polynomial in the imaginary part. Thus in notation such as $(1 + |t|)^B$, the B is some absolute constant—whose value is not important—which may vary from line to line.

Assuming Theorem 4, we can now estimate Ξ_2. By the symmetry of the zeros, it is enough to consider those in the first quadrant. Subdividing the region $[\lambda^{-1}, 1/2] \times \mathbf{R}^+$ into small squares of side λ^{-1}, namely,

$$R(m, n) = \left[\frac{m}{\lambda}, \frac{m+1}{\lambda}\right] \times \left[\frac{n}{\lambda}, \frac{n+1}{\lambda}\right]$$

with $1 \leqslant m \leqslant \lambda$, $0 \leqslant n$, we estimate the contribution Ξ_2^1 of those zeros:

$$\begin{aligned}
\sum_{f \in S_2(q)^*} \operatorname{Re}\big(\Xi_2^1(f, \lambda)\big) &\leqslant \lambda \sum_{m=1}^{\lambda} \sum_{n \geqslant 0} N\left(f; \frac{1}{2} + \frac{n}{\lambda}, \frac{m}{\lambda}, \frac{m+1}{\lambda}\right) \sup_{s \in R(m,n)} \big|\hat{\psi}(\lambda s)\big| \\
&\ll \lambda \sum_{m=1}^{\lambda} \sum_{n \geqslant 0} \left(1 + \frac{n+1}{\lambda}\right)^A q^{1-c(n/\lambda)} (\log q) \lambda^{-1} e^{m+1} (1+n)^{-k} \\
&\ll q(\log q)
\end{aligned}$$

if we choose $\theta < c$ and $k \geqslant A + 3$. Hence, from (35), dividing out by λ, we deduce Theorem 2.

4.2. Reduction to second moment estimates. Theorem 4 is the analogue of a result of Selberg [Sel, Theorem 4] for Dirichlet characters. We borrow the general principle from that paper (with some simplifications also found in [L]), starting with a crucial lemma that reduces the theorem to some estimates of a mollified second moment of values of $L(f,s)$, $f \in S_2(q)^*$.

LEMMA 8 (Selberg, [Sel, Lemma 14]). *Let h be a function holomorphic in the region*

$$\{s \in \mathbf{C} \mid \mathrm{Re}(s) \geqslant \alpha,\ t_1 \leqslant \mathrm{Im}(s) \leqslant t_2\}$$

satisfying

$$h(s) = 1 + o\left(\exp\left(-\frac{\pi}{t_2 - t_1}\mathrm{Re}(s)\right)\right) \tag{38}$$

in this region, uniformly as $\mathrm{Re}(s) \to +\infty$. *Denoting the zeros of f (in the interior of this region) by $\rho = \beta + i\gamma$, we have*

$$\begin{aligned} 2(t_2 - t_1)\sum_{\rho} \sin\left(\pi\frac{\gamma - t_1}{t_2 - t_1}\right)\sinh\left(\pi\frac{\beta-\alpha}{t_2 - t_1}\right) \\ = \int_{t_1}^{t_2} \sin\left(\pi\frac{t - t_1}{t_2 - t_1}\right)\log\left|h(\alpha + it)\right| dt \\ + \int_{\alpha}^{+\infty} \sinh\left(\pi\frac{\sigma - \alpha}{t_2 - t_1}\right)\left(\log\left|h(\sigma + it_1)\right| + \log\left|h(\sigma + it_2)\right|\right) d\sigma \end{aligned}$$

(where the zeros are also summed with multiplicity).

We apply this lemma to the functions $1 - (M(f,s)L(f,s) - 1)^2$, where $M(f,s)$ is a suitable mollifier for which (38) holds, for α equal to $1/2 + (\log q)^{-1}$. This means that $M(f,s)$ must approximate quite closely the inverse of $L(f,s)$.

LEMMA 9. *The inverse $L(f,s)^{-1}$ is given by the Dirichlet series*

$$L(f,s)^{-1} = \sum_{m,n \geqslant 1} \varepsilon_q(n)\mu(m)\mu(mn)^2\lambda_f(m)(mn^2)^{-s},$$

which is absolutely convergent for $\mathrm{Re}(s) > 1$.

Proof. This is an immediate consequence of the Euler product expansion

$$L(f,s)^{-1} = \prod_p \left(1 - \lambda_f(p)p^{-s} + \varepsilon_q(p)p^{-2s}\right)$$

by multiplicativity. (Every integer $\ell \geqslant 1$ has a unique expression as $\ell = mn^2 r$ with m, n, r coprime in pairs, m and n square-free and r cube-full.) □

We also define, for every $M \geqslant 1$, a continuous function g_M by

$$g_M(x) = \begin{cases} 1, & \text{if } x \leqslant \sqrt{M}, \\ \dfrac{\log M/x}{\log \sqrt{M}}, & \text{if } \sqrt{M} \leqslant x \leqslant M, \\ 0, & \text{if } x > M. \end{cases} \tag{39}$$

Then for M fixed and any integer $1 \leqslant m \leqslant M$, we let

$$x_m(s) = \frac{\mu(m)}{m^{s-1/2}} \sum_{n \geqslant 1} \frac{\varepsilon_q(n)\mu(mn)^2}{n^{2s}} g_M(mn) \tag{40}$$

and we define the mollifier

$$M(f,s) = \sum_{m \leqslant M} \frac{x_m(s)}{\sqrt{m}} \lambda_f(m). \tag{41}$$

In the context of $\mathrm{GL}(2)$, mollifiers were first used by Hafner [Haf].

We observe that $M(f,s)$ is a Dirichlet polynomial of length at most M, with coefficients

$$c_f(\ell) = \sum_{mn^2 = \ell} \varepsilon_q(n)\mu(m)\mu(mn)^2 \lambda_f(m) g_M(mn), \tag{42}$$

and by Deligne's bound, they are bounded by

$$\left| c_f(\ell) \right| \leqslant \sum_{m \mid \ell} \tau(m) \leqslant \tau(\ell)^2. \tag{43}$$

From the definition, it follows easily that for $M = q^{\Delta}$ with $\Delta > 0$, we have

$$M(f,s)L(f,s) = 1 + O\left((\log q)^{15} q^{\Delta(1-\sigma)/2} \right) \tag{44}$$

uniformly for $\mathrm{Re}(s) = \sigma \to +\infty$.

The density theorem follows from a good estimate for the average of the second moment of $M(f,s)L(f,s)$, $\mathrm{Re}(s) \geqslant 1/2 + (\log q)^{-1}$.

Proposition 4. *Let $M = q^{\Delta}$ with $\Delta < 1/4$, and let c be any positive real number with $c < \Delta$. Then there exists a constant $B > 0$ such that for all q prime large enough,*

$$\sum_{f \in S_2(q)^*} \left| M(f, \beta + it) L(f, \beta + it) - 1 \right|^2 \ll \left(1 + |t|\right)^B q^{1 - c(\beta - 1/2)} \tag{45}$$

uniformly for $\beta \geqslant 1/2 + (\log q)^{-1}$ and $t \in \mathbf{R}$, the implied constant depending only on Δ and c.

Assuming this proposition, the proof of Theorem 4 is completed as in [Sel] and [K, 5.2] by an application of Lemma 8 to the functions $1 - (M(f,s)L(f,s) - 1)^2$.

4.3. The harmonic second moment. Proposition 4 is proved by the method of Section 3, going through a corresponding weighted result first.

PROPOSITION 5. *Let $M = q^{\Delta}$ with $\Delta < 1/4$ and let $\beta = 1/2 + b(\log q)^{-1}$, where $b > 0$ is any constant. For all q prime large enough, we have*

$$\sum_{f \in S_2(q)^*}^{h} \left| M(f, \beta + it) L(f, \beta + it) \right|^2 \ll \left(1 + |t|\right)^B \tag{46}$$

for some absolute constant $B > 0$. The implied constant depends only on b and Δ.

We write $\beta = 1/2 + \delta$ and assume only $\delta = b(\log q)^{-1}$. Then for simplicity, we define

$$M_2(\delta) = \sum_{f \in S_2(q)^*}^{h} \left| M(f, \beta + it) L(f, \beta + it) \right|^2, \tag{47}$$

which we consider as a quadratic form in the coefficients $x_m = x_m(\beta + it)$ of the mollifier. To emphasize this viewpoint, it is convenient to simply write x_m and $M(f)$ while performing transformations to facilitate the ultimate estimations.

Let $f \in S_2(q)^*$ and $\beta = 1/2 + \delta$ with $0 < \delta < 1/2$ be given. Choose an integer $N \geqslant 1$ (which has to be large enough; $N = 2$ works already) and a real polynomial G satisfying

$$G(-s) = G(s), \tag{48}$$

$$G(-N) = \cdots = G(-1) = 0 \tag{49}$$

and having no zeros for $-1/2 \leqslant \mathrm{Re}(s) \leqslant 1/2$.

Let $t \in \mathbf{R}$ be a fixed real number. Define the entire function $Z(f, s)$ by

$$Z(f, s) = \Lambda\left(f, s + \frac{1}{2} + it\right) \Lambda\left(f, s + \frac{1}{2} - it\right),$$

which satisfies the functional equation

$$Z(f, s) = Z(f, -s).$$

Since the Fourier coefficients $\lambda_f(n)$ of f are real, we have

$$\left| \Lambda(f, \beta + it) \right|^2 = Z(f, \beta). \tag{50}$$

We now consider the complex integral

$$\begin{aligned} I_\delta &= \frac{1}{2i\pi} \int_{(2)} Z(f, s) G(s + it) G(s - it) \frac{ds}{s - \delta} \\ &= \frac{1}{2i\pi} \int_{(2)} L\left(f, s + \frac{1}{2} + it\right) L\left(f, s + \frac{1}{2} - it\right) H(s) \left(\frac{\sqrt{q}}{2\pi}\right)^{2s+1} \frac{ds}{s - \delta} \end{aligned}$$

(defined, as a function of δ, for all $\delta \in \mathbf{R}$) with

$$H(s) = G(s+it)G(s-it)\Gamma(s+1+it)\Gamma(s+1-it).$$

This integral is absolutely convergent (by Stirling's formula). From (49), zeros of the polynomial G cancel the first poles of the Γ function, so H is holomorphic for $\mathrm{Re}(s) > -N-1$. Next, normalizing G, we can assume that $H(\delta) = 1$. The gamma function has exponential decay in vertical strips, while G has polynomial growth; in fact, by Stirling's formula, H satisfies (uniformly in vertical strips)

$$H(s) \ll \big(1+|t|+|\mathrm{Im}(s)|\big)^B e^{-\pi|\mathrm{Im}(s)|}, \quad \text{for some constant } B > 0.$$

We can shift the contour of integration to the line $\mathrm{Re}(s) = -2$; only a simple pole at $s = \delta$ appears while shifting, with residue

$$\mathrm{Res}_{s=\delta} Z(f,s)G(s+it)G(s-it)\frac{1}{s-\delta} = \Big(\frac{q}{4\pi^2}\Big)^\beta H(\delta)\big|L(f,\beta+it)\big|^2$$

by (50).

On the line $\mathrm{Re}(s) = -2$, the integral is seen to be

$$\frac{1}{2i\pi}\int_{(-2)} Z(f,s)G(s+it)G(s-it)\frac{ds}{s-\delta} = -I_{-\delta}$$

by the change of variable $s \mapsto -s$, using the functional equation of $Z(f,s)$ and the symmetry $G(s) = G(-s)$. Hence we have the formula

$$\Big(\frac{q}{4\pi^2}\Big)^\beta H(\delta)\big|L(f,\beta+it)\big|^2 = I_\delta + I_{-\delta}. \tag{51}$$

On the other hand, using the Hecke relation (6), one has, in the region of absolute convergence, the identity

$$L(f,s+it)L(f,s-it) = \zeta_q(2s)\sum_{n\geqslant 1}\lambda_f(n)\eta_t(n)n^{-s},$$

where

$$\eta_t(n) = \sum_{ab=n}\Big(\frac{a}{b}\Big)^{it}. \tag{52}$$

We may then integrate term-by-term to obtain

$$I_\delta = \Big(\frac{q}{4\pi^2}\Big)^{1/2}\sum_n \frac{\lambda_f(n)}{n^{1/2}}\,\eta_t(n)W_\delta\Big(\frac{4\pi^2 n}{q}\Big)$$

where

$$W_\delta(y)=\frac{1}{2i\pi}\int_{(2)} H(s)\zeta_q(1+2s)y^{-s}\,\frac{ds}{s-\delta}. \tag{53}$$

Finally, we get

$$\left(\frac{q}{4\pi^2}\right)^{\delta} H(\delta)\left|L(f,\beta+it)\right|^2=\sum_{n\geqslant 1}\frac{\lambda_f(n)}{\sqrt{n}}\,\eta_t(n)U\left(\frac{4\pi^2 n}{q}\right) \tag{54}$$

where

$$U(y)=W_\delta(y)+W_{-\delta}(y)=\frac{1}{2i\pi}\int_{(2)} H(s)\zeta_q(1+2s)y^{-s}\,\frac{2s\,ds}{(s-\delta)(s+\delta)}. \tag{55}$$

We conclude this section by listing the basic properties of the test function U and the arithmetic function η_t. These should be skipped and consulted when referred to later.

LEMMA 10. *For $\delta\neq 0$, we have*

$$U(y)=H(\delta)\zeta_q(1+2\delta)y^{-\delta}+H(-\delta)\zeta_q(1-2\delta)y^{\delta}+O\big(y^N\big(1+|t|\big)^B\big) \tag{56}$$

for $0\leqslant y\leqslant 1$, and

$$U(y)\ll_j y^{-j}\big(1+|t|\big)^B \quad \textit{for all } j\geqslant 1 \tag{57}$$

for $y\geqslant 1$, B depending on j.

Proof. This follows easily by shifting the line of integration either to the right to $\mathrm{Re}(s)=j$ or to the left to $\mathrm{Re}(s)=-N-\delta$ and then computing the residues. □

LEMMA 11. *For all $t\in\mathbf{R}$, the arithmetic function η_t is real valued. It satisfies the identities*

$$\eta_t(n)\eta_t(m)=\sum_{d\mid(n,m)}\eta_t\Big(\frac{nm}{d^2}\Big), \tag{58}$$

$$\eta_t(nm)=\sum_{d\mid(n,m)}\mu(d)\eta_t\Big(\frac{n}{d}\Big)\eta_t\Big(\frac{m}{d}\Big), \tag{59}$$

$$\sum_{n\geqslant 1}\eta_t(n)n^{-s}=\zeta(s-it)\zeta(s+it), \tag{60}$$

$$\sum_{n\geqslant 1}\eta_t(n)^2 n^{-s}=\frac{\zeta(s-2it)\zeta(s)^2\zeta(s+2it)}{\zeta(2s)}, \tag{61}$$

$$\sum_{n\geqslant 1}\eta_t(n^2)n^{-s}=\frac{\zeta(s-2it)\zeta(s)\zeta(s+2it)}{\zeta(2s)}, \tag{62}$$

and the estimate

$$|\eta_t(n)| \leqslant \tau(n). \tag{63}$$

Proof. Everything can be checked elementarily by direct computations, but it may as well be deduced from the fact that $\eta_t(n)$ is a Hecke eigenvalue for the operator $T(n)$ acting on the derivative at $s = 1/2$ of the nonholomorphic Eisenstein series $E(z,s)$ of level 1. □

We now come to the mollifier $M(f)$. By multiplicativity of the coefficients $\lambda_f(n)$, once more we have

$$|M(f)|^2 = \sum_b \frac{1}{b} \sum_{m_1,m_2} \frac{\lambda_f(m_1m_2)}{\sqrt{m_1m_2}} x_{bm_1}\overline{x_{bm_2}}$$

so that, by (54), the second moment $M_2(\delta)$ is given by

$$\left(\frac{q}{4\pi^2}\right)^\delta H(\delta)M_2(\delta) = \sum_b \frac{1}{b} \sum_{n\geqslant 1} \sum_{m_1,m_2} \frac{\eta_t(n)}{\sqrt{m_1m_2n}} x_{bm_1}\overline{x_{bm_2}}\, U\left(\frac{4\pi^2 n}{q}\right) \Delta(m_1m_2, n)$$

where Δ is the Delta symbol. We recall that (by (12))

$$\Delta(m,n) = \delta(m,n) + O\big((mn)^{1/2}(\log q)^2 q^{-3/2}\big)$$

for $m, n \leqslant q$, where the implied constant is absolute.

Using (40) to estimate x_m, we have

$$x_m \ll \zeta(1+2\delta)m^{-\delta};$$

the contribution to $M_2(\delta)$ of the remainder term of $\Delta(m,n)$ is at most

$$\frac{(\log q)^2}{q^{3/2}} \sum_b \frac{1}{b} \left| \sum_{bm\leqslant M} \tau(m)x_{bm} \right|^2 \left| \sum_{n\geqslant 1} U\left(\frac{4\pi^2 n}{q}\right) \right| \ll_\epsilon \big(1+|t|\big)^B (qM)^\epsilon \frac{M^2}{q^{1/2}}. \tag{64}$$

We now study the diagonal contribution where $n = m_1m_2$, namely, the sum $M'(\delta)$ defined by the equality

$$\left(\frac{q}{4\pi^2}\right)^\delta H(\delta)M'(\delta) = \sum_b \frac{1}{b} \sum_{m_1,m_2} \frac{\eta_t(m_1m_2)}{m_1m_2} x_{bm_1}\overline{x_{bm_2}}\, U\left(\frac{4\pi^2 m_1m_2}{q}\right).$$

Inserting (56), we have

$$\left(\frac{q}{4\pi^2}\right)^\delta H(\delta)M'(\delta) = \left(\frac{q}{4\pi^2}\right)^\delta H(\delta)M''(\delta) + O_\epsilon\Big(\big(1+|t|\big)^B (qM)^\epsilon q^{-1/2} M^2\Big) \tag{65}$$

where the sum $M''(\delta)$ is given by

$$\begin{aligned}\left(\frac{q}{4\pi^2}\right)^{\delta} & H(\delta)M''(\delta)\\ &= \left(\frac{q}{4\pi^2}\right)^{\delta} H(\delta)\zeta_q(1+2\delta)\sum_b \frac{1}{b}\sum_{m_1,m_2}\frac{\eta_t(m_1m_2)}{(m_1m_2)^{1+\delta}}\, x_{bm_1}\overline{x_{bm_2}} \\ &\quad + \left(\frac{q}{4\pi^2}\right)^{-\delta} H(-\delta)\zeta_q(1-2\delta)\sum_b \frac{1}{b}\sum_{m_1,m_2}\frac{\eta_t(m_1m_2)}{(m_1m_2)^{1-\delta}}\, x_{bm_1}\overline{x_{bm_2}}\end{aligned} \tag{66}$$

and the error term is estimated by

$$\big(1+|t|\big)^B \frac{1}{\sqrt{q}}\sum_{b\leqslant M}\frac{1}{b}\left|\sum_{bm\leqslant M}\frac{\tau(m)}{\sqrt{m}}\, x_{bm}\right|^2 \ll \big(1+|t|\big)^B (qM)^{\epsilon} q^{-1/2} M^2.$$

4.4. Diagonalization and estimation of the second moment. First, m_1 and m_2 can be separated in (66) by means of the Möbius inversion formula (59), so that

$$\begin{aligned}\left(\frac{q}{4\pi^2}\right)^{\delta} & H(\delta)M''(\delta)\\ &= \left(\frac{q}{4\pi^2}\right)^{\delta} H(\delta)\zeta_q(1+2\delta)\sum_b \frac{1}{b}\sum_a \frac{\mu(a)}{a^{2(1+\delta)}}\sum_{m_1,m_2}\frac{\eta_t(m_1)\eta_t(m_2)}{(m_1m_2)^{1+\delta}}\, x_{abm_1}\overline{x_{abm_2}} \\ &\quad + \left(\frac{q}{4\pi^2}\right)^{-\delta} H(-\delta)\zeta_q(1-2\delta)\sum_b \frac{1}{b}\sum_a \frac{\mu(a)}{a^{2(1-\delta)}}\sum_{m_1,m_2}\frac{\eta_t(m_1)\eta_t(m_2)}{(m_1m_2)^{1-\delta}}\, x_{abm_1}\overline{x_{abm_2}},\end{aligned}$$

and we can collect the single variable $k = ab$, introducing the arithmetic function

$$\nu_\delta(k) = \sum_{ab=k}\frac{\mu(a)}{a^{1+2\delta}}$$

to derive

$$\begin{aligned}\left(\frac{q}{4\pi^2}\right)^{\delta} H(\delta)M''(\delta) &= \left(\frac{q}{4\pi^2}\right)^{\delta} H(\delta)\zeta_q(1+2\delta)\sum_k \frac{\nu_\delta(k)}{k}\left|\sum_m \frac{\eta_t(m)}{m^{1+\delta}}\, x_{km}\right|^2 \\ &\quad + \left(\frac{q}{4\pi^2}\right)^{-\delta} H(-\delta)\zeta_q(1-2\delta)\sum_k \frac{\nu_{-\delta}(k)}{k}\left|\sum_m \frac{\eta_t(m)}{m^{1-\delta}}\, x_{km}\right|^2.\end{aligned} \tag{67}$$

Following Selberg, we observe that for $0 < \delta < 1/2$, the inequalities

$$-\zeta_q(1-2\delta) \geqslant 0, \qquad H(-\delta) = \big|\Gamma(1-\delta+it)G(-\delta+it)\big|^2 > 0,$$
$$\nu_{-\delta}(k) = \prod_{p|k}\big(1-p^{-1+2\delta}\big) \geqslant 0$$

hold. Hence, by positivity,

$$M''(\delta) \leqslant \zeta_q(1+2\delta) \sum_k \frac{\nu_\delta(k)}{k} \left| \sum_m \frac{\eta_t(m)}{m^{1+\delta}} x_{km} \right|^2. \tag{68}$$

Let

$$y_k = \sum_m \frac{\eta_t(m)}{m^{1+\delta}} x_{km} \tag{69}$$

(which is supported on square-free integers $k \leqslant M$).

Proposition 6. *Assume $\delta = b(\log q)^{-1}$ for some (absolute) constant $b > 0$ and $M = q^\Delta$ with $\Delta < 1/4$. Then for k square-free, $k \leqslant M$, we have*

$$k^{\delta+it}\xi(k) y_k \ll \frac{1}{\log q}$$

where

$$\xi(k) = \prod_{p|k} \left(1 - p^{-1/2}\right).$$

Remark. This saving of a factor $\log q$ is the critical moment. It comes essentially from cancellation due to the oscillations of the Möbius function, or in other words, from the prime number theorem.

Proof. We proceed as in [L]. From the definition (40), for $s = \beta + it = 1/2 + \delta + it$, we have

$$x_{km} = \frac{\mu(k)}{k^{\delta+it}} \times \frac{\mu(m)}{m^{\delta+it}} \sum_n \frac{\mu(kmn)^2}{n^{1+2\delta+2it}} g_M(kmn)$$

(there is no $\varepsilon_q(n)$ since $n \leqslant kmn \leqslant M < q$). Therefore,

$$y_k = \frac{\mu(k)}{k^{\delta+it}} \sum_{m,n} \frac{\mu(kmn)^2 \mu(m) \eta_t(m) n^{-it}}{(mn)^{1+2\delta+it}} g_M(kmn).$$

Assume first that $1 \leqslant k \leqslant \sqrt{M}$ (and, of course, k is square-free). We use the following integral formula: for all $\ell \geqslant 1$,

$$g_M(k\ell) = \frac{1}{2i\pi} \int_{(2)} \frac{\left(\sqrt{M}/k\right)^s \left(M^{s/2} - 1\right)}{\log \sqrt{M}} \ell^{-s} \frac{ds}{s^2}, \tag{70}$$

which follows from

$$\frac{1}{2i\pi} \int_{(2)} y^s \frac{ds}{s^2} = \begin{cases} \log y, & \text{if } y \geqslant 1, \\ 0, & \text{if } 0 < y \leqslant 1. \end{cases}$$

Hence

$$k^{\delta+it}y_k = \frac{1}{2i\pi}\int_{(2)} L_k(s+1+2\delta+it)\frac{\left(\sqrt{M}/k\right)^s\left(M^{s/2}-1\right)}{\log\sqrt{M}}\,\frac{ds}{s^2} \tag{71}$$

with the ad hoc Dirichlet series

$$L_k(s) = \sum_{\ell\geqslant 1}\mu(k\ell)^2\left(\sum_{mn=\ell}\mu(m)\eta_t(m)n^{-it}\right)\ell^{-s},$$

which is easily computed. Indeed, the inner sum is the coefficient of ℓ^{-s} in the product

$$\begin{aligned}\zeta(s+it)\sum_{m\geqslant 1}\mu(m)\eta_t(m)m^{-s} &= \prod_p\left(1-p^{-s-it}\right)^{-1}\left(1-p^{-s}\left(p^{it}+p^{-it}\right)\right)\\ &\qquad\qquad\text{(by multiplicativity and the definition of } \eta_t)\\ &= \prod_p\left(1-p^{-s+it}\left(1-p^{-s-it}\right)^{-1}\right)\\ &= \prod_p\left(1-p^{-s+it}\sum_{j\geqslant 0}p^{-j(s+it)}\right)\end{aligned}$$

and $L_k(s)$ is obtained from this Dirichlet series by taking the subseries restricted to integers prime to k and square-free (this is the effect of inserting $\mu(k\ell)^2$ in a Dirichlet series). This gives the very simple answer

$$L_k(s) = \zeta_k(s-it)^{-1}.$$

From the theorems of Hadamard and de la Vallée-Poussin, $\zeta(s)$ has no zeros on the line $\mathrm{Re}(s)=1$, and, more precisely, the estimate

$$\zeta(s)^{-1} \ll \log\left(2+|\mathrm{Im}(s)|\right) \tag{72}$$

holds with an absolute implied constant (see [Tit, Ch. 3]) uniformly for

$$\mathrm{Re}(s) \geqslant 1-\frac{D}{\log\left(2+|\mathrm{Im}(s)|\right)}$$

($D>0$ being another absolute constant).

Let r be small enough so that the circle $|s|\leqslant r$ is included in this zero-free region, and let $0<r<1/2$ (of course, any $r<1/2$ will do, the Riemann hypothesis being numerically valid in such a range). In (71), we shift the integration to the contour C consisting of the vertical line $\mathrm{Re}(s)=0$ from $-i\infty$ to $-ir$, followed by the half-circle $s=re(x)$ for $-\pi/2\leqslant x\leqslant \pi/2$, and then again the line $\mathrm{Re}(s)=0$ from ir to $i\infty$. By the choice of r, this is permissible; the contour shift passes through a unique simple pole at $s=0$ (simple because of the zero of $s\mapsto M^{s/2}-1$). Thus from the formula for $L_k(s)$, we get

$$k^{\delta+it}y_k=\zeta_k(1+2\delta)^{-1}+\frac{1}{2i\pi}\int_C \frac{\zeta(s+1+2\delta)^{-1}}{\prod_{p|k}\left(1-p^{-(s+1+2\delta)}\right)}\,\frac{\left(\sqrt{M}/k\right)^s\left(M^{s/2}-1\right)}{\log\sqrt{M}}\,\frac{ds}{s^2}.$$

The integral over C is now estimated. Using (72), the part from ir to $i\infty$ is dominated by

$$\frac{1}{\log M}\left|\int_r^{+\infty}\frac{\zeta(1+2\delta+iu)^{-1}}{\prod_{p|k}\left(1-p^{-(1+2\delta+iu)}\right)}\left(\frac{\sqrt{M}}{k}\right)^{iu}\left(M^{iu/2}-1\right)\frac{du}{u^2}\right|\ll\frac{1}{\log q}\,\frac{1}{\xi(k)},$$

since clearly

$$\prod_{p|k}\left(1-p^{-1-2\delta}\right)^{-1}\leqslant\xi(k)^{-1}.$$

The same holds without change for the other vertical half-line. For the semicircle, we use the fact that $k\leqslant\sqrt{M}$ so that

$$\left(\frac{\sqrt{M}}{k}\right)^s\left(M^{s/2}-1\right)\ll 1$$

on this semicircle where $\mathrm{Re}(s)<0$, and similarly, the product over primes dividing k is dominated by its value at $s=-r$, which is

$$\prod_{p|k}\left(1-p^{r-1-\delta}\right)^{-1}\leqslant\xi(k)^{-1}$$

since $r<1/2$. Hence the same bound holds again. Now we collect the results and use the assumption that $\delta=b(\log q)^{-1}$, which implies

$$\zeta(1+2\delta)^{-1}\ll(\log q)^{-1}.$$

Hence for $k\leqslant\sqrt{M}$, we immediately obtain the desired bound

$$\xi(k)k^{1+\delta}y_k\ll\frac{1}{\log q}.$$

This is also true in the case $\sqrt{M}\leqslant k\leqslant M$: we use a similar reasoning, replacing (70) by the other formula,

$$g_M(k\ell)=\frac{1}{2i\pi}\int_{(2)}\frac{(M/k)^s}{\log\sqrt{M}}\,\ell^{-s}\,\frac{ds}{s^2},$$

and using the same contour shift. This finishes the proof. □

From Proposition 6 and (68), we have

$$M''(\delta) \leqslant \zeta_q(1+2\delta) \sum_{k \leqslant M} \frac{\nu_\delta(k)}{k} |y_k|^2$$
$$\ll \frac{\zeta_q(1+2\delta)}{(\log q)^2} \sum_{k \leqslant M} \frac{\mu(k)^2 \nu_\delta(k)}{\xi(k)^2} k^{-(1+2\delta)}$$
$$\ll \frac{1}{\log q} \sum_{k \leqslant M} \frac{\mu(k)^2 \nu_\delta(k)}{\xi(k)^2} k^{-(1+2\delta)}.$$

Now the last sum above is a partial sum for a Dirichlet series admitting analytic continuation to $\mathrm{Re}(s) \geqslant 7/8$ with a simple pole as $s = 1$ and therefore

$$\sum_{k \leqslant M} \frac{\mu(k)^2}{\xi(k)^2} k^{-(1+2\delta)} \ll \log q.$$

To conclude the proof of Proposition 5, we look back to the error terms (64) and (65) introduced in going from the original second moment $M_2(\delta)$ to $M'(\delta)$ and then to $M''(\delta)$, and we see that they bring a total contribution that is

$$\ll q^{-\gamma} \big(1 + |t|\big)^B$$

for some $\gamma = \gamma(\Delta) > 0$ if $M = q^\Delta$ with $\Delta < 1/4$.

4.5. Harmonic average and natural average, II. Having estimated the harmonic average $A^h[|M(f, \beta+it)L(f, \beta+it)|^2]$, we now apply Proposition 2 to study

$$A\Big[\big|M(f, \beta+it)L(f, \beta+it)\big|^2\Big].$$

The notation and assumptions are the same as at the beginning of Section 4.3: recall that $\beta = 1/2 + \delta$, and $M = q^\Delta$ with $\Delta < 1/4$.

First we check the conditions: (22) is contained in Proposition 5, while for (23), we have the following lemma.

LEMMA 12. *For all $f \in S_2(q)^*$, it holds that*

$$\omega_f \big|M(f, \beta+it)L(f, \beta+it)\big|^2 \ll q^{-1/4}\big(1+|t|\big)^2$$

for all β with $\beta \geqslant 1/2$, the implied constant being absolute.

Proof. Using (43), the trivial bound for $M(f, \beta+it)$ is

$$M(f, \beta+it) \ll \sqrt{M}(\log q)^3,$$

while the convexity bound for $L(f, s)$ on the critical line gives

$$L(f, \beta+it) \ll_\varepsilon q^{1/4+\varepsilon}\big(1+|t|\big)^{1/2+\varepsilon}$$

for $\beta \geqslant 1/2$. Since, on the other hand, we have $\omega_f \ll (\log q)q^{-1}$ from (20), the result follows. $\square$

Hence Proposition 2 with $x = q^\kappa$, for any $\kappa > 0$, gives the equality

$$A\Big[\big|M(f,\beta+it)L(f,\beta+it)\big|^2\Big] = \frac{\dim J_0(q)}{\zeta(2)} A^h\Big[\omega_f(x)\big|M(f,\beta+it)L(f,\beta+it)\big|^2\Big] + O\big((1+|t|)^B q^{1-\gamma}\big) \tag{73}$$

for some $\gamma = \gamma(\Delta,\kappa) > 0$. (The dependence in t of the error term has to be checked by looking back at the proof of the proposition.)

We let

$$\begin{aligned}\mathcal{M}_2(\delta) &= A^h\Big[\omega_f(x)\big|M(f,\beta+it)L(f,\beta+it)\big|^2\Big] \\ &= \sum_{d\ell^2 \leqslant x} \frac{1}{d\ell^2} \sum_{f\in S_2(q)^*}^h \lambda_f\big(d^2\big)\big|M(f,\beta+it)L(f,\beta+it)\big|^2.\end{aligned}$$

Computing as before, we get

$$\left(\frac{q}{4\pi^2}\right)^\delta H(\delta)\mathcal{M}_2(\delta) = \sum_b \frac{1}{b}\sum_{n\geqslant 1}\sum_{m_1,m_2} \frac{\eta_t(n)}{\sqrt{m_1m_2n}} x_{bm_1}\overline{x_{bm_2}} U\left(\frac{4\pi^2 n}{q}\right)\Delta^n(m_1m_2,n)$$

where we put

$$\begin{aligned}\Delta^n(m,n) &= \sum_{\ell\leqslant x} A^h\big[\rho_f(\ell)\lambda_f(m)\lambda_f(n)\big] && (74)\\ &= \sum_{d\ell^2\leqslant x}\ \sum_{f\in S_2(q)^*}^h \lambda_f\big(d^2\big)\lambda_f(m)\lambda_f(n) \\ &= \sum_{d\ell^2\leqslant x}\frac{1}{d\ell^2}\sum_{r\mid(d^2,m)} \delta\left(\frac{md^2}{r^2},n\right) + O\left((\log q)^3\frac{x\sqrt{mn}}{q^{3/2}}\right) && (75)\end{aligned}$$

by (12) again. The error term yields a contribution which, by the same computation as in (64), is at most

$$\ll_\epsilon (qMx)^\epsilon \frac{xM^{2(1-\delta)}}{\sqrt{q}}\big(1+|t|\big)^B \ll_\kappa \big(1+|t|\big)^B q^{-\gamma} \tag{76}$$

for some $\gamma > 0$, if κ is taken small enough.

The diagonal contribution $n = md^2r^{-2}$ is

$$\sum_b \frac{1}{b}\sum_{m_1,m_2}\frac{x_{bm_1}\overline{x_{bm_2}}}{m_1m_2}\sum_{d\ell^2\leqslant x}(d\ell)^{-2}\sum_{r\mid(m_1m_2,d^2)} r\eta_t\left(\frac{m_1m_2d^2}{r^2}\right)U\left(\frac{4\pi^2m_1m_2d^2}{qr^2}\right),$$

and we use (56) to get

$$\Big(\frac{q}{4\pi^2}\Big)^{\delta} H(\delta)\mathcal{M}_2(\delta) = \Big(\frac{q}{4\pi^2}\Big)^{\delta} H(\delta)\zeta_q(1+2\delta)\mathcal{M}(\delta) + \Big(\frac{q}{4\pi^2}\Big)^{-\delta} H(-\delta)\zeta_q(1-2\delta)\mathcal{M}(-\delta) + O\big(\big(1+|t|\big)^B q^{-\gamma}\big) \tag{77}$$

for some $\gamma > 0$, where the sum $\mathcal{M}(\delta)$ is

$$\mathcal{M}(\delta) = \sum_b \frac{1}{b} \sum_{m_1,m_2} \frac{x_{bm_1}\overline{x_{bm_2}}}{(m_1m_2)^{1+\delta}} \sum_{d\ell^2\leqslant x} \frac{1}{d^{2+2\delta}\ell^2} \sum_{r\mid(m_1m_2,d^2)} r^{1+2\delta}\eta_t\Big(\frac{m_1m_2d^2}{r^2}\Big). \tag{78}$$

We first compute the inner sum, showing in particular that we can now again extend the summation over d, ℓ to all integers, and then we compute this complete series.

We define a function $u(s,r)$ for $s \in \mathbf{C}$, $r \geqslant 1$ an integer, by

$$u(s,r) = \sum_{ab=r} \mu(a)b^s = \prod_{p\mid r}\big(p^s - 1\big),$$

and a function $v_x(s,r)$, supported on cube-free integers r, by

$$v_x(s,r) = \sum_{\substack{d\ell^2\leqslant x\\ r\mid d^2}} \ell^{-2}d^{-2s}\eta_t\Big(\frac{d^2}{r}\Big). \tag{79}$$

We also denote by $v(s,r)$ the function obtained by removing the constraint $d\ell^2 \leqslant x$ in the definition of $v_x(s,r)$.

From (59), we have for every integer m and n,

$$\sum_{r\mid(m,n)} r^s\eta_t\Big(\frac{mn}{r^2}\Big) = \sum_{r\mid(m,n)} u(s,r)\eta_t\Big(\frac{m}{r}\Big)\eta_t\Big(\frac{n}{r}\Big);$$

hence

$$\sum_{d\ell^2\leqslant x} \frac{1}{d^{2+2\delta}\ell^2} \sum_{r\mid(m_1m_2,d^2)} r^{1+2\delta}\eta_t\Big(\frac{m_1m_2d^2}{r^2}\Big) = \sum_{r\mid m_1m_2} \eta_t\Big(\frac{m_1m_2}{r}\Big)u(1+2\delta,r)v_x(1+\delta,r). \tag{80}$$

We define two multiplicative functions N and M by

$$N(r) = \prod_{p\mid r} p, \qquad M(r) = \prod_{p\parallel r} p.$$

Lemma 13. *For all cube-free integers $r \geqslant 1$, and s with* $\mathrm{Re}(s) = \sigma > 1/2$, *we have*

$$v_x(s,r) = v(s,r) + O\left(\frac{(\log x)^3 \tau(r)}{N(r)^{2\sigma-1/2}\sqrt{x}}\right). \tag{81}$$

Moreover,

$$v(s,1) = \frac{\zeta(2)\zeta(2s)\zeta(2s+2it)\zeta(2s-2it)}{\zeta(4s)},$$

and for all $r \geqslant 1$,

$$v(s,r) = v(s,1)N(r)^{-2s} \prod_{p \| r} \frac{\eta_t(p)}{1+p^{-2s}}.$$

Proof. The point is that for a cube-free integer r and any $d \geqslant 1$, we have $r \mid d^2$ if and only if $N(r) \mid d$. Since

$$r = M(r)\,\frac{N(r)^2}{M(r)^2} = \frac{N(r)^2}{M(r)},$$

we can write

$$v_x(s,r) = \sum_{\substack{d\ell^2 \leqslant x \\ N(r)|d}} \ell^{-2} d^{-2s} \eta_t\left(\frac{d^2}{r}\right) = N(r)^{-2s} \sum_{d\ell^2 \leqslant x/N(r)} \ell^{-2} d^{-2s} \eta_t\big(M(r)d^2\big)$$

and proceed similarly without constraint for $v(s,r)$. Now, putting $y = x/N(r)$,

$$\sum_{d\ell^2 > x/N(r)} \ell^{-2} d^{-2s} \eta_t\big(M(r)d^2\big) \ll \tau\big(M(r)\big)\left(\sum_{\ell^2<y} \ell^{-2} \sum_{d>y/\ell^2} \tau\big(d^2\big)d^{-2\sigma} + \sum_{\ell^2>y} \ell^{-2}\right)$$
$$\ll \tau(r)(\log x)^3 y^{-1/2},$$

and this gives the first formula.

To compute $v(s,r)$ (which is a kind of nonprimitive symmetric square for η_t), we define

$$v'(s,r) = \sum_{d \geqslant 1} \eta_t\big(M(r)d^2\big)d^{-2s}$$

so that $v(s,r) = \zeta(2)N(r)^{-2s}v'(s,r)$. We denote by $Z(s)$ the full symmetric square given by (62), and by Z_p its p-factor.

Every integer d has a unique expression $d = d_1 d_2$ with $d_1 \mid M(r)^\infty$ and $(d_2, M(r)) = 1$; so by multiplicativity, we get

$$\begin{aligned} v'(s,r) &= \left(\sum_{(d,M(r))=1} \eta_t\big(d^2\big)d^{-2s}\right)\left(\sum_{d|M(r)^\infty} \eta_t\big(M(r)d^2\big)d^{-2s}\right) \\ &= Z(2s)\prod_{p\|r} Z_p(2s)^{-1} \times \prod_{p\|r} \sum_{k\geqslant 0} \eta_t\big(p^{2k+1}\big)p^{-2ks}. \end{aligned}$$

Again by multiplicativity,

$$\eta_t\left(p^{2k+1}\right)=\eta_t(p)\eta_t\left(p^{2k}\right)-\eta_t\left(p^{2k-1}\right)$$

for $k \geqslant 1$, so that

$$\left(1+p^{-2s}\right)\sum_{k\geqslant 0}\eta_t\left(p^{2k+1}\right)p^{-2ks}=\eta_t(p)Z_p(2s),$$

which yields

$$v'(s,r)=Z(2s)\prod_{p\| r}\frac{\eta_t(p)}{1+p^{-2s}}.$$

This proves the lemma, since

$$v(s,1)=Z(2s)=\frac{\zeta(2)\zeta(2s)\zeta(2s+2it)\zeta(2s-2it)}{\zeta(4s)}$$

from (62). □

Now let $w_x(s,m)$ be the function defined for $s\in\mathbf{C}$ and $m\geqslant 1$ by

$$w_x(s,r)=\sum_{r|m}\eta_t\left(\frac{m}{r}\right)u(2s-1,r)v_x(s,r), \tag{82}$$

and let $w(s,m)$ be the same with $v(s,r)$ replacing $v_x(s,r)$. Then from (80) and (78) comes the formula

$$\mathcal{M}(\delta)=\sum_b\frac{1}{b}\sum_{m_1,m_2}\frac{w_x(1+\delta,m)}{(m_1m_2)^{1+\delta}}\,x_{bm_1}\overline{x_{bm_2}}. \tag{83}$$

LEMMA 14. *Assume that $\delta=b(\log q)^{-1}$ for some constant $b>0$. Then*

$$\mathcal{M}(\delta)=\sum_b\frac{1}{b}\sum_{m_1,m_2}\frac{w(1+\delta,m_1m_2)}{(m_1m_2)^{1+\delta}}\,x_{bm_1}\overline{x_{bm_2}}+O\left(q^{-\gamma}\right)$$

for some $\gamma=\gamma(\kappa,\Delta)>0$.

Proof. Since m_1 and m_2 are square-free, the product m_1m_2 and any divisor thereof are always cube-free. So we use (81) to replace $v_x(1+\delta,r)$ by $v(1+\delta,r)$. This first gives

$$w(1+\delta,m)=w_x(1+\delta,m)+O\left(\frac{\tau(m)^3(\log x)^3}{\sqrt{x}}\right),$$

because the error term is bounded by

$$\sum_{r|m}\tau\left(\frac{m}{r}\right)\left|u(1+2\delta,r)\right|\frac{(\log x)^3\tau(r)}{N(r)^{3/2+2\delta}\sqrt{x}}\ll\frac{\tau(m)^3(\log x)^3}{\sqrt{x}}$$

by the estimate

$$|u(1+2\delta,r)| = \left|\prod_{p|r}\left(p^{1+2\delta}-1\right)\right| \leqslant N(r)^{1+2\delta}.$$

Then inserting this inside $\mathcal{M}(\delta)$ gives the result. □

4.6. End of evaluation of the second moment. After Lemma 14, we now want to diagonalize the quadratic form $\mathcal{M}(\delta)$ (more precisely, its main term). This is done by the following transformations (compare [KM3] and [K], where more complicated expressions involving functions that are less multiplicative than v occur).

We have seen that $v(1+\delta,r)$ is the product of a constant and a multiplicative function, and by Dirichlet convolution, it follows that $w(1+\delta,m)$ is also

$$w(1+\delta,m) = v(1+\delta,1)\overline{w}(m)$$

with $\overline{w}$ multiplicative.

We extract the common divisor of m_1 and m_2 and remove the ensuing coprimality condition by Möbius inversion:

$$\begin{aligned}
&\sum_b \frac{1}{b}\sum_{m_1,m_2} \frac{w(1+\delta,m_1m_2)}{(m_1m_2)^{1+\delta}}\, x_{bm_1}\overline{x_{bm_2}} \\
&\quad = v(1+\delta,1)\sum_b \frac{1}{b}\sum_{m_1,m_2}\frac{\overline{w}(m_1m_2)}{(m_1m_2)^{1+\delta}}\, x_{bm_1}\overline{x_{bm_2}} \\
&\quad = v(1+\delta,1)\sum_b \frac{1}{b}\sum_a \frac{\overline{w}\left(a^2\right)}{a^{2(1+\delta)}}\sum_{(m_1,m_2)=1}\frac{\overline{w}(m_1)\overline{w}(m_2)}{(m_1m_2)^{1+\delta}}\, x_{abm_1}\overline{x_{abm_2}} \\
&\quad = v(1+\delta,1)\sum_b \frac{1}{b}\sum_a \frac{\overline{w}\left(a^2\right)}{a^{2(1+\delta)}}\sum_d \frac{\mu(d)\overline{w}(d)^2}{d^{2(1+\delta)}}\sum_{m_1,m_2}\frac{\overline{w}(m_1)\overline{w}(m_2)}{(m_1m_2)^{1+\delta}}\, x_{adbm_1}\overline{x_{adbm_2}} \\
&\quad = v(1+\delta,1)\sum_k \frac{\tilde{v}_\delta(k)}{k}\left|\sum_m \frac{\overline{w}(m)}{m^{1+\delta}}\, x_{km}\right|^2
\end{aligned} \tag{84}$$

with

$$\tilde{v}_\delta(k) = \sum_{abd=k}\frac{\mu(d)\overline{w}(d)^2\overline{w}\left(a^2\right)}{(ad)^{1+2\delta}}.$$

Remember that $\tilde{v}_\delta(k)$ also depends on t (through η_t involved in $\overline{w}$).

LEMMA 15. *There exists an absolute constant $b>0$ such that if $|\delta| \leqslant b(\log q)^{-1}$, then*

$$\tilde{v}_\delta(k) \geqslant 0$$

for all $t \in \mathbf{R}$ *and all* $k < q$, *and*

$$v(1+\delta,1) \gg 1$$

for all $t \in \mathbf{R}$.

Proof. By multiplicativity, it is enough to consider $k = p$ prime and $p < q$. Then $e^{-2b} \leqslant p^{2\delta} \leqslant e^{2b}$. We have

$$\nu_\delta(p) = 1 + \frac{1}{p^{1+2\delta}}\bigl(\overline{w}\bigl(p^2\bigr) - \overline{w}(p)^2\bigr)$$

and by direct computation, from Lemma 13 and the definition of $w(1+\delta, m)$,

$$\begin{aligned}\overline{w}(p) &= \eta_t(p) + \bigl(p^{1+2\delta}-1\bigr)p^{-2(1+\delta)}\frac{\eta_t(p)}{1+p^{-2(1+\delta)}}\\ &= \eta_t(p)\frac{p^{2+2\delta}+p^{1+2\delta}}{p^{2+2\delta}+1}\end{aligned}$$

and similarly,

$$\begin{aligned}\overline{w}\bigl(p^2\bigr) &= \eta_t\bigl(p^2\bigr) + \eta_t(p)^2\frac{p^{1+2\delta}-1}{p^{2+2\delta}+1} + \frac{p^{1+2\delta}-1}{p^{2+2\delta}}\\ &= \eta_t(p)^2\frac{p^{2+2\delta}+p^{1+2\delta}}{p^{2+2\delta}+1} - 1 + \frac{p^{1+2\delta}-1}{p^{2+2\delta}}.\end{aligned}$$

Hence

$$\overline{w}\bigl(p^2\bigr) - \overline{w}(p)^2 = -\eta_t(p)^2\,\frac{p^{2+2\delta}+p^{1+2\delta}}{p^{2+2\delta}+1}\,\frac{p^{1+2\delta}-1}{p^{2+2\delta}+1} - 1 + \frac{p^{1+2\delta}-1}{p^{2+2\delta}}.$$

For p large enough, the result is now clear uniformly in δ, and we leave to the reader the choice of argument for dealing with small primes. For instance, note that for $p = 2$, $\delta = 0$, we obtain

$$\overline{w}\bigl(p^2\bigr) - \overline{w}(p)^2 \geqslant -4\,\frac{6}{25} - 1 + \frac{1}{4} > -2,$$

and the result is now clear by continuity. □

We can now conclude.

PROPOSITION 7. *Assume that* $\delta = b(\log q)^{-1}$ *with* $b > 0$, *a fixed constant such that the previous lemma applies. Then for* $M = q^{\Delta}$ *with* $\Delta < 1/4$, *we have*

$$\sum_{f \in S_2(q)^*} \bigl|M(f,\beta+it)L(f,\beta+it)\bigr|^2 \ll q\bigl(1+|t|\bigr)^B$$

for some absolute constant $B \geqslant 0$. *The implied constant depends only on* Δ.

Proof. From Lemma 15, the computation of $\mathcal{M}(\delta)$, and the subsequent diagonalization of the main term, we see that for q large enough, we have

$$\mathcal{M}(-\delta) \geqslant 0.$$

Hence, using the same trick as before, that $\zeta_q(1-2\delta) \leqslant 0$, we get, by positivity, the inequality

$$\mathcal{M}_2(\delta) \leqslant \zeta(1+2\delta)\mathcal{M}(\delta) \ll v(1+\delta,1)\zeta(1+2\delta)\sum_k \frac{\tilde{\nu}_\delta(k)}{k}\left|\sum_m \frac{\overline{w}(m)}{m^{1+\delta}}x_{km}\right|^2.$$

Now, in terms of the linear forms y_k introduced in (69), we can write

$$\sum_m \frac{\overline{w}(m)}{m^{1+\delta}}x_{km} = \sum_{a,b}\frac{\eta_t(a)\eta_t(b)u(1+\delta,b)}{(ab)^{1+\delta}\big(b^{2(1+\delta)}+1\big)}x_{abk} = \sum_b \frac{u(1+\delta,b)\eta_t(b)}{b^{1+\delta}\big(b^{2(1+\delta)}+1\big)}y_{bk},$$

since for square-free n we have $N(n)=n$. But $u(1+\delta,b)\leqslant b^{1+2\delta}$ for b square-free, and Proposition 6 immediately gives

$$\sum_m \frac{\overline{w}(m)}{m^{1+\delta}}x_{km} \ll \frac{1}{k^{(1+\delta)}\xi(k)}\,\frac{1}{\log q}.$$

Then, going back to (73) to conclude, the proof is completed as for the harmonic average. □

Proposition 4 is an easy consequence of this estimate near the critical line. Indeed, it first immediately provides the bound

$$\sum_{f\in S_2(q)^*}\big|M(f,\beta+it)L(f,\beta+it)-1\big|^2 \ll q\big(1+|t|\big)^B \tag{85}$$

for $\beta = 1/2+b(\log q)^{-1}$. On the other hand, for $\mathrm{Re}(s)=\sigma>1$, we have

$$\sum_{f\in S_2(q)^*}\big|M(f,s)L(f,s)-1\big|^2 \ll q^{1-\Delta(1-\sigma)}(\log q)^{30} \tag{86}$$

as a consequence of the trivial individual bound of Lemma 44.

Finally, by means of (85), (86), and the convexity principle of Phragmen-Lindelöf (for subharmonic functions), Proposition 4 follows.

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