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THE BOCHNER-RIESZ CONJECTURE IMPLIES THE RESTRICTION CONJECTURE

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1. Introduction Fix $n \geqslant 2$. For any $1 \leqslant p \leqslant \infty$ and $\alpha \geqslant 0$, we use $BR(p, \alpha)$ to denote that $S^{\delta(p)+\alpha}$ is bounded on L^p, where $\delta(p) = \max(n|1/p - 1/2| - 1/2, 0)$ and S^δ is the Bochner-Riesz multiplier

$$\widehat{S^\delta f}(\xi) = (1 - |\xi|^2)_+^\delta \hat{f}(\xi).$$

The Bochner-Riesz conjecture is the statement that $BR(p, \varepsilon)$ holds for all $1 \leqslant p \leqslant \infty$ and $\varepsilon > 0$. Apart from the trivial estimate $BR(2, 0)$, this conjecture is optimal (see [12]).

Similarly, we use $R(p, \alpha)$ to denote the localized restriction estimate

$$\|\Re f\|_{L^p(S^{n-1})} \lesssim R^\alpha \|f\|_{L^p(B(0,R))}$$

for f supported in $B(0, R)$, where $\Re f = \hat{f}|_{S^{n-1}}$ is the sphere restriction operator. Note that $R(p, 0)$ is equivalent to the global restriction estimate

$$\|\Re f\|_{L^p(S^{n-1})} \lesssim \|f\|_{L^p(\mathbf{R}^n)}.$$

The restriction conjecture[1] asserts that $R(p, 0)$ holds for all $1 \leqslant p < 2n/(n+1)$. Localized restriction theorems such as the L^2-estimate $R(2, 1/2)$ have been used before (see [1], [2]) as a stepping stone to global restriction theorems.

Although no formal equivalence has been proven between the two conjectures, they are widely believed to be at least heuristically equivalent. For example, the implication Restriction $\Rightarrow$ Bochner-Riesz is known for the parabolic analogue of the spherical problem (see Carbery [3]), or when the (p, p) restriction hypothesis is strengthened to a $(p, 2)$-estimate (see, e.g., Fefferman [11], Christ [8], [6], [5], Tao [24]). Table 1 illustrates the close relationship between progress on the two conjectures.

In this paper we present a formal proof of the implication in the reverse direction, that the Bochner-Riesz conjecture implies the restriction conjecture, as an immediate consequence of the following two results.

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[1] There is a slight strengthening of this conjecture with imbalanced exponents, but we do not concern ourselves with that here.

TABLE 1
Progress on Bochner-Riesz and restriction for $n = 3$

$BR(p,\varepsilon)$ when	Reference	$R(p,0)$ when	Reference
$p' = \infty$	Bochner	$p' = \infty$	Riemann-Lebesgue
$p' \geqslant 6$	Fefferman, 1970 [11]	$p' \geqslant 6$	Stein, 1967
$p' \geqslant 4$	Tomas, 1975 [26]	$p' > 4$	Tomas, 1975 [26]
		$p' \geqslant 4$	Stein, 1975
$p' \geqslant 4 - 8/75$	Bourgain, 1991 [1]		
$p' \geqslant 4 - 2/15$	Bourgain, 1995 [2]	$p' > 4 - 2/15$	Bourgain, 1991 [1]
$p' \geqslant 4 - 2/11$	Wolff, 1995 [28]	$p' > 4 - 2/11$	Wolff, 1995 [28]
$p' = 3$?	(critical value)	$p' > 3$?	(critical value)

THEOREM 1.1. *If $1 \leqslant p \leqslant 2n/(n+1)$, then $BR(p,\alpha) \Rightarrow R(p,2\alpha)$.*

THEOREM 1.2. *If $1 < p < 2$ and $0 < \alpha \ll 1$, then $R(p,\alpha) \Rightarrow R(q,0)$ whenever*

$$\frac{1}{q} > \frac{1}{p} + \frac{C}{\log \frac{1}{\alpha}}.$$

COROLLARY 1.3. *If $1 < p \leqslant 2n/(n+1)$ and $BR(p,\varepsilon)$ holds for all $\varepsilon > 0$, then $R(q,0)$ holds for all $1 \leqslant q < p$.*

It is well known that $R(p,0)$ is false for any $p \geqslant 2n/(n+1)$, so the exponent in the above corollary is optimal.

The first theorem is based on the very simple observation, that the Bochner-Riesz operator resembles the restriction operator when evaluated at points far away from the support of the function. The factor of 2 in the conclusion is a familiar aspect of the numerology of the problem and reflects the principle that the local behaviour at scale R is best studied at a global scale of R^2. A related "osculation" argument has appeared in [27].

The second theorem is more involved than the first and can be viewed as a statement: If one can control the (L^p, L^p)-norm of $\mathfrak{R}$ reasonably well on large balls, then one can control the (L^q, L^p)-norm of the same operator on all of $\mathbf{R}^n$, where q is a slightly worse exponent than p. The proof of the theorem requires two ideas: first, the observation that control of the restriction operator on large balls can be bootstrapped to control of the same operator on "sparse" unions of these large balls; second, an "exponentially low-dimensional" Calderón-Zygmund decomposition that covers a set E by a small number of sparse collections of balls that are not too huge.

In both theorems the uncertainty principle plays a minor but recurring role. For example, we take advantage of the frequency localization of the restriction operator to introduce a spatial uncertainty of $O(1)$, and conversely we use a spatial localization at scale R to introduce a frequency uncertainty of $O(1/R)$.

We also rely on the freedom to introduce an uncertainty of $O(1)$ to the phase function of an oscillatory integral. One concrete statement of this principle is formalized as a stability lemma in Lemma 2.1.

Theorem 1.2 may be compared with the following result of the same type, which is due to Bourgain (see also [17]).

THEOREM 1.4 [2]. *If* $p \leqslant 2(n+1)/(n+3)$, $0 \leqslant \alpha \leqslant (n+1)/2p'$, *then* $R(p,\alpha) \Rightarrow R(q,0)$ *whenever*

$$q' \geqslant 2 + \frac{1}{(n+1)/2p' - \alpha}.$$

The theorem of Bourgain relies on a modification of the L^2-restriction theory of Tomas and Stein [26] and is more efficient than Theorem 1.2 for most values of α; however, for our purposes, Theorem 1.2 is superior because it does not lose any exponents in the limit $\alpha \to 0$.

In the last section, we present some extensions and related results to the main theorem; for instance, we show the equivalence of the Nikodym and Kakeya maximal operator conjectures. We will develop these results in a more general setting in a subsequent paper [25].

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2. Proof of Theorem 1.1. Let f be a compactly supported function. Fix $p \leqslant 2n/(n+1)$ so that $\delta(p) = n(1/p - 1/2) - 1/2$. The basic idea of the argument is to exploit the heuristic approximation

$$S^{\delta(p)+\alpha} f(x) \sim |x|^{-\alpha} \frac{e^{\pm 2\pi i |x|}}{|x|^{n/p}} \hat{f}\left(\pm \frac{x}{|x|}\right),$$

which is accurate when x is very far away from the support of f. One may see this informally by using a stationary phase to estimate the kernel of the Bochner-Riesz multiplier and then using a Taylor series to linearize the phase function.

Proof. We now begin the rigorous proof. By the standard Carleson-Sjölin reduction (see, e.g., [22], [10], [4]), $BR(p,\alpha)$ implies the estimate

$$\|G_R f\|_p \lesssim R^{-n/p'} R^{\alpha} \|f\|_p,$$

where

$$G_R f(x) = \int e^{2\pi i R|x-y|} \psi(x,y) f(y)\, dy,$$

and $\psi(x, y)$ is a bump function supported on the set $|x|, |y| \lesssim 1, |x - y| \sim 1$. This estimate can be deduced from $BR(p, \alpha)$ by applying a partition of unity of the sphere to the multiplier for $S^{\delta(p)+\alpha}$, using the stationary-phase asymptotics for the resulting kernel, localizing to the scale R, and rescaling; see also the remarks at the end of this section.

If we replace R by R^2 in the above estimate, perform the rescaling $\tilde{y} = y/R$, and restrict $x, \tilde{y}$ to the range $|x| \sim 1$, $|\tilde{y}| \lesssim 1$, we obtain

$$\|T_R g(x)\|_p \lesssim R^{-n/p'} R^{2\alpha} \|g\|_p \tag{1}$$

for all g supported in the unit ball, where

$$T_R g(x) = \int e^{2\pi i |R^2 x - R\tilde{y}|} \tilde{\psi}(x, \tilde{y}) g(\tilde{y})\, d\tilde{y},$$

and $\tilde{\psi}$ is a bump function supported in the region $|x| \sim 1, |\tilde{y}| \lesssim 1$.

We now use the Taylor series approximation

$$|R^2 x - R\tilde{y}| = R^2|x| - R\left\langle \frac{x}{|x|}, \tilde{y} \right\rangle + a(x, \tilde{y}, 1/R),$$

where $a(x, \tilde{y}, \tau)$ is a bounded function that is C^∞ in x and τ for $|x| \sim 1$, $0 \leqslant \tau \lesssim 1$, $|\tilde{y}| \lesssim 1$. The $a(x, \tilde{y}, 1/R)$ error term in the phase can be dropped, as can the $\tilde{y}$-dependence of the amplitude function $\tilde{\psi}(x, \tilde{y})$ (see the remarks at the end of this section), so that T_R can be replaced by $\tilde{T}_R$ in (1), where

$$\tilde{T}_R g(x) = \psi(x) e^{2\pi i R^2 |x|} \int e^{-2\pi i R \langle x/|x|, \tilde{y} \rangle} g(\tilde{y})\, d\tilde{y} = \psi(x) e^{2\pi i R^2 |x|} \hat{g}\left(R \frac{x}{|x|}\right),$$

and ψ is a function supported on $\{|x| \sim 1\}$. The conclusion $R(p, 2\alpha)$ then follows by rescaling. □

We now justify the heuristic principle, used repeatedly in the above argument: one can freely modify the amplitude or phase of a compactly supported oscillatory integral by a smooth $O(1)$ factor without affecting its regularity properties. Variations of this stability lemma have appeared elsewhere in the literature; see, for example, Lemma 2.10 of Christ [7] or the argument in [15].

Lemma 2.1. *For each $R \gg 1$, let $K_R(x, y)$ be a bounded, compactly supported function, where the bounds and support are independent of R, and let $1 \leqslant p, q \leqslant \infty$. Suppose $b(x, y, \tau)$ is a bounded function that is C^∞ in x and τ, for $0 \leqslant \tau \lesssim 1$, and x, y in the support of K. Then, if the operators A_R and B_R are defined by*

$$A_R f(x) = \int K_R(x, y) f(y)\, dy,$$

$$B_R f(x) = \int K_R(x, y) b(x, y, 1/R) f(y)\, dy,$$

then for all $N > 0$ *we have*

$$\|B_R\|_{L^p \to L^q} \lesssim \|A_R\|_{L^p \to L^q} + O(R^{-N}).$$

In particular, if $b = e^{ia}$ for some $a(x, y, \tau)$, then the operator norms of A_R and B_R are comparable (except for a rapidly decreasing error).

Proof. Since x, y are compactly supported, one can break b up into a Fourier series in x and a Taylor series in τ as

$$b(x, y, \tau) = \sum_{j=0}^{N} \sum_{k \in \mathbf{Z}^n} c_{j,k} \tau^j e^{2\pi i \langle k, x \rangle} f_{j,k}(y) + O(\tau^N),$$

where the $f_{j,k}$ are uniformly bounded functions, the $c_{j,k}$ are constants that are rapidly decreasing in k, $\mathbf{Z}^n$ is a suitable lattice, and N is an arbitrarily large number. The result then follows from the operator identity

$$B_R = \sum_{j=0}^{N} \sum_{k \in \mathbf{Z}^n} c_{j,k} R^{-j} e^{2\pi i \langle k, \cdot \rangle} A_R f_{j,k} + O(R^{-N})$$

and the triangle inequality. □

Note that the L^p-spaces in the above lemma can easily be replaced by Lorentz spaces, Orlicz spaces, or any other quasi-normed spaces that are stable under multiplication by L^∞-functions. The requirement that K is compactly supported in y can also be weakened. The fact that no regularity is assumed in y, although not needed in the proof of the above theorem, is useful in other applications, such as replacing a continuous estimate by its discrete analogue or vice versa. (See, e.g., Bourgain [2] for examples of such heuristics.)

3. Proof of Theorem 1.2. The first step is to bootstrap the localized restriction estimate so that it applies to functions that are supported on a sparse union of balls of constant radius. The idea is to exploit the fact that the Fourier transforms of functions on widely separated balls behave in a quasi-orthogonal manner on the unit sphere.

Definition 3.1. A collection $\{B(x_i, R)\}_{i=1}^N$ of R-balls is *sparse* if the centers x_i are $R^C N^C$ separated. (C denotes a constant that changes from line to line.)

LEMMA 3.2. *Suppose* $R(p, \alpha)$ *holds. Then*

$$\|\Re f\|_p \lesssim R^\alpha \|f\|_p$$

whenever f *is supported on* $\bigcup_i B(x_i, R)$ *and* $\{B(x_i, R)\}$ *is a sparse collection of balls.*

Proof. We start by modifying the localized restriction hypothesis slightly. Let $\tilde{\mathfrak{R}}f$ be the restriction of $\hat{f}$ to the annulus A_R of thickness $\sim 1/R$ around the unit sphere. By averaging the restriction operator over all dilations with dilation factor $1 + O(1/R)$, we see that $R(p, \alpha)$ implies the estimate

$$\|\tilde{\mathfrak{R}}f\|_{L^p(A_R)} \lesssim R^{-1/p} R^{\alpha} \|f\|_p \tag{2}$$

whenever f is supported on $B(0, R)$. By translation symmetry, we see that this estimate is also valid if f is supported in any other ball $B(x, R)$ of radius R.

Write $f = \sum_i f_i \varphi_i$, where each f_i is supported on $B(x_i, R)$, $\varphi_i(x) = R^{-n}\varphi((x - x_i)/R)$, and φ is a Schwartz function that is positive on the unit ball and whose Fourier transform is supported on the unit ball. Since $\|f\|_p \sim (\sum_i \|f_i\|_p^p)^{1/p}$ and $\mathfrak{R}f = \sum_i \tilde{\mathfrak{R}}f_i * \hat{\varphi}_i|_{S^{n-1}}$, the lemma follows from (2) if we can prove the estimate

$$\left\| \sum_i F_i * \hat{\varphi}_i|_{S^{n-1}} \right\|_p \lesssim R^{1/p} \left(\sum_i \|F_i\|_p^p \right)^{1/p}$$

for all $F_i \in L^p(\mathbf{R}^n)$.

When $p = 1$, this estimate follows from replacing everything by absolute values and using Fubini's theorem, so it suffices by real interpolation to prove the estimate for $p = 2$. By Plancherel's theorem, this estimate is equivalent to the inequality

$$\left\| \mathfrak{R}\left(\sum_i f_i \varphi_i \right) \right\|_2 \lesssim R^{1/2} \left(\sum_i \|f_i\|_2^2 \right)^{1/2}$$

for all $f_i \in L^2(\mathbf{R}^n)$. By translation symmetry and the rapid decrease of φ, we may replace φ_i by the characteristic function χ_i of $B(x_i, R)$. By the T^*T method, we can then replace the inequality with the equivalent inequality

$$\left(\sum_j \left\| \chi_j \mathfrak{R}^* \mathfrak{R}\left(\sum_i f_i \chi_i \right) \right\|_2^2 \right)^{1/2} \lesssim R \left(\sum_i \|f_i\|_2^2 \right)^{1/2}.$$

By Schur's test and self-adjointness, this estimate follows from

$$\sup_j \sum_i \|\chi_j \mathfrak{R}^* \mathfrak{R} \chi_i\| \lesssim R,$$

where $\|T\|$ denotes the L^2-operator norm of T.

This, in turn, follows from the estimates

$$\|\chi_i \mathfrak{R}^* \mathfrak{R} \chi_i\| \lesssim R$$

and

$$\|\chi_j \mathfrak{R}^* \mathfrak{R} \chi_i\| \lesssim R^{-C} N^{-C}, \quad j \neq i.$$

The former estimate follows from the localized restriction theorem $R(2, 1/2)$; this estimate can be proven by introducing a frequency uncertainty of $1/R$, as permitted by the uncertainty principle (or Lemma 2.1) and using Plancherel's theorem. The latter estimate follows from Schur's test and the observation that the kernel of $\chi_j \mathfrak{R}^* \mathfrak{R} \chi_i$ is $O(|x_i - x_j|^{-(n-1)/2}) = O(R^{-C} N^{-C})$. This, in turn, follows from the standard decay estimates of the Fourier transform of surface measure on the sphere. Details may be found in [26] or [2]. □

Fix $p < 2$ and $\alpha > 0$, and assume that $R(p, \alpha)$ holds. To prove the global restriction theorem, we start with some standard reductions. First, it suffices to show the Lorentz space estimate

$$\|\mathfrak{R} f\|_p \lesssim \|f\|_{q_0, 1}, \tag{3}$$

where $1/q_0 = 1/p + C/\log(1/\alpha)$. This implies the theorem by Hölder's inequality and interpolation with known restriction theorems, such as the trivial (L^1, L^∞)-estimate.

By averaging over translations, it suffices to show (3) when f is a measure supported on a discrete lattice $\mathbf{Z}^n$ and the $L^{q_0,1}$-norm is replaced by the discrete norm $l^{q_0,1}$. We may then replace f by $f * \chi$ (and revert to the continuous norm $L^{q_0,1}$), where χ is the characteristic function of the cube of size c and $c \sim 1$ is chosen so that $\hat{\chi}$ is positive on the unit sphere. Combining the two reductions, we see that it suffices to verify the original estimate (3) when f is constant on c-cubes.

Since we are working in $L^{q_0,1}$, we may take $f = \chi_E$ for some set E, which we can assume to be the union of c-cubes. Our objective is now to prove that

$$\|\mathfrak{R} \chi_E\|_p \leqslant C_\alpha |E|^{1/p + C/\log(1/\alpha)}. \tag{4}$$

This can be accomplished with the aid of the following Calderón-Zygmund–type lemma, which covers such a set E by a reasonably small number of sparse collections of balls, where one has some modest control on the size of the balls.

LEMMA 3.3. *Suppose E is the union of c-cubes and $N \geqslant 1$. Then there exist $O(N|E|^{1/N})$ sparse collections of balls that cover E, such that the balls in each collection have radius $O(|E|^{C^N})$.*

By combining this lemma with Lemma 3.2 and the triangle inequality, we obtain

$$\|\mathfrak{R}\chi_E\|_p \lesssim N|E|^{1/N}(|E|^{C^N})^\alpha |E|^{1/p},$$

and (4) follows by setting $N = C^{-1}\log(1/\alpha)$ for some sufficiently large C.

Proof of Lemma 3.2. Define the radii R_k for $0 \leqslant k \leqslant N$ by $R_0 = 1$, $R_{k+1} = |E|^C R_k^C$; note that $R_k = O(|E|^{C^k})$ for each k. Starting with $k = 1$ and proceeding recursively, we set E_k to be the set of all $x \in E$ that are not in any E_j for $j < k$ and are such that

$$|E \cap B(x, R_k)| \leqslant |E|^{k/N}.$$

For every $1 \leqslant k \leqslant N$ and $x \in E_k$, we have by construction and hypothesis that

$$|E \cap B(x, R_{k-1})| \gtrsim |E|^{(k-1)/N}.$$

Thus for every $x \in E_k$, the set $E_k \cap B(x, R_k)$ can be covered by $O(|E|^{1/N})$ R_{k-1}-balls. This implies that the entire set E_k can be covered by $O(|E|^{1/N})$ collections of R_{k-1}-balls that are R_k-separated. Since the cardinality of these collections can be at most $O(|E|)$, we see from the construction of the R_k that the collections are sparse. The lemma then follows by noting that $E = \bigcup_{k=1}^N E_k$. □

4. Additional remarks. Several variations on the above theme are possible. Further results on these problems will be presented in future papers (e.g., [25]).

In this section we discuss some open conjectures related to spherical Bochner-Riesz and spherical restriction conjectures, which we describe briefly below for reference. (The notation and wording of the conjecture may differ slightly from the standard formulation.)

CONJECTURE 4.1 (Local smoothing for the wave equation [21]). *If $u(x,t)$ is a solution of the homogeneous Cauchy problem*

$$\left(-\frac{\partial^2}{\partial t^2} + \Delta\right) u(x,t) = 0,$$

$$u(x,0) = f(x), \qquad u_t(x,0) = 0,$$

then one has

$$\|u\|_{L^p(\mathbf{R}^n \times [1,2])} \lesssim \|(1+\sqrt{-\Delta})^\varepsilon f\|_p$$

for all $\varepsilon > 0$, where $p = 2n/(n-1)$ and Δ is the Laplacian.

CONJECTURE 4.2 (Maximal spherical Bochner-Riesz [23], [5]). *For $R, \delta > 0$ define the operators S_R^δ by*

$$\widehat{S_R^\delta f}(\xi) = \left(1 - \left|\frac{\xi}{R}\right|^2\right)_+^\delta \hat{f}(\xi).$$

Then for all $2 \leqslant p \leqslant \infty$ and $\delta > \delta(p)$, one has the estimate

$$\left\|\sup_{R>0} |S_R^\delta f|\right\|_p \lesssim \|f\|_p.$$

CONJECTURE 4.3 (Parabolic Bochner-Riesz [3]). *For $\delta > 0$ define the operators $\tilde{S}^\delta$ by*

$$\widehat{\tilde{S}_R^\delta f}(\underline{\xi}, \xi_n) = \eta(\xi)(\xi_n - |\underline{\xi}|^2)_+^\delta \hat{f}(\underline{\xi}, \xi_n),$$

for $(\underline{\xi}, \xi_n) \in \mathbf{R}^{n-1} \times \mathbf{R}$, where η is smooth and compactly supported. Then $\tilde{S}^\delta$ is bounded on L^p whenever $1 \leqslant p \leqslant \infty$ and $\delta > \delta(p)$.

CONJECTURE 4.4 (Parabolic restriction [3], [17]). *Let $S = \{(\underline{\xi}, |\underline{\xi}|^2) : |\underline{\xi}| \lesssim 1\}$ be a subset of the standard paraboloid. Then for all $1 \leqslant p < 2n/(n-1)$, we have*

$$\|\hat{f}|_S\|_{L^p(S)} \lesssim \|f\|_p.$$

CONJECTURE 4.5 (Kakeya maximal function [1]). *If f is a compactly supported function and $0 < \delta \ll 1$, we define the Kakeya maximal function f_δ^* on S^{n-1} by*

$$f_\delta^*(\omega) = \sup_{T \| \omega} \frac{1}{|T|} \int_T f(x)\, dx,$$

where the supremum is taken over all $1 \times \delta$-tubes T that are oriented in the direction ω. Let $K(p, \alpha)$ denote the estimate

$$\|f_\delta^*\|_p \lesssim \delta^{(n/p)-1-\alpha} \|f\|_p.$$

Then $K(p, \alpha)$ holds for all $1 \leqslant p \leqslant n$ and $\alpha > 0$.

CONJECTURE 4.6 (Kakeya set [1]). *Define a Kakeya set to be any subset E of $\mathbf{R}^n$ that contains a unit line segment in each direction. Then all Kakeya sets have Minkowski dimension n.*

CONJECTURE 4.7 (Nikodym maximal function [1]). *If f is a compactly supported function and $0 < \delta \ll 1$, we define the Kakeya maximal function f_δ^* on $\mathbf{R}^n$ by*

$$f_\delta^*(x) = \sup_{x \in T} \frac{1}{|T|} \int_T f(y)\, dy,$$

where the supremum is taken over all $1 \times \delta$-tubes T that contain x. Let $N(p, \alpha)$ denote the estimate

$$\|f_\delta^{**}\|_p \lesssim \delta^{(n/p)-1-\alpha} \|f\|_p.$$

Then $N(p, \alpha)$ holds for all $1 \leqslant p \leqslant n$ and $\alpha > 0$.

CONJECTURE 4.8 (Nikodym set [1]). *Define a Nikodym set to be any subset E of $\mathbf{R}^n$ such that for every $x \in \mathbf{R}^n$ there exists a line l through x such that $E \cap l$ contains a unit line segment. Then all Nikodym sets have Minkowski dimension n.*

It is known (see [13], [10]) that both the spherical and parabolic restriction conjectures imply the Kakeya maximal conjecture and that (see [1]) the Kakeya (resp., Nikodym) maximal function conjecture implies the Kakeya (resp., Nikodym) set conjecture. Also, the local smoothing conjecture of Sogge [21] is known to imply the maximal Bochner-Riesz conjecture and hence the Bochner-Riesz conjecture. Finally, the parabolic restriction conjecture is known to imply the parabolic Bochner-Riesz conjecture (see [3]). These equivalences, together with the ones discussed in this paper, are summarized in Table 2.

The arguments of Sections 2 and 3 can be generalized to arbitrary oscillatory integrals. Informally, the result is that any oscillatory integral estimate can be

TABLE 2
Known implications between Bochner-Riesz, Kakeya, and related conjectures

	local smoothing	
	⇓	
	maximal Bochner-Riesz	
	⇓	
	spherical Bochner-Riesz	
	⇓	
	spherical restriction	
	⇓	
parabolic restriction	⟺	parabolic Bochner-Riesz
	⇓	
maximal Kakeya	⟺	maximal Nikodym
	⇓	
	Kakeya set	
	⇓	
	Nikodym set	

linearized into a global restriction theorem. More precisely, suppose

$$S_R f(x) = \int e^{-2\pi i R\phi(x,y)} \psi(x,y) f(y)\, dy$$

is a family of oscillatory integrals, where the amplitude ψ is smooth and compactly supported in $\mathbf{R}^n \times \mathbf{R}^m$, where $m-1 \leqslant n \leqslant m$ and ϕ is a phase function of constant rank $m-1$ in the sense of Hörmander [14]. Then for each point y_0 in the domain of y, one can associate a Fourier restriction operator $\mathfrak{R}_{y_0}$ associated to the linearization of the S_R at y_0:

$$\mathfrak{R}_{y_0} f(x) = \psi(x, y_0) \hat{f}(\nabla_y \phi(x, y_0)).$$

THEOREM 4.9. *If S_R is bounded from $L^p(\mathbf{R}^m)$ to $L^p(\mathbf{R}^n)$ with operator norm $O_\varepsilon(R^{-m/p'+\varepsilon})$ for all $\varepsilon > 0$, then for each y_0 and all $1 \leqslant q < p$, the operators $\mathfrak{R}_{y_0}$ are bounded from $L^q(\mathbf{R}^m)$ to $L^q(\mathbf{R}^n)$.*

Proof. The proof of the theorem consists of routine modifications of the arguments given in Sections 2 and 3. Note that the hypothesis of Conjecture 4.2 implies that the restriction surface $\{\nabla_y \phi(x, y_0) : x \in \mathbf{R}^n\}$ cannot degenerate to infinite order because of the Knapp counterexample. Hence one can use the stationary phase to obtain some decay to Fourier transforms of measures on this surface, so that the argument in Lemma 3.2 carries over. □

As an example of this theorem, we may reverse the results of Carbery [3] and conclude that the Bochner-Riesz and restriction conjectures for paraboloids are equivalent. Another example (see [16]) comes from the Bochner-Riesz oscillatory integrals S_λ arising from a Riemannian manifold M^n:

$$S_\lambda f(x) = \int_{M^n} e^{i\lambda \operatorname{dist}(x,y)} a(x,y) f(y)\, dy.$$

Any positive (L^p, L^p)-result on the Hörmander problem for this operator implies a corresponding result for the spherical restriction operator, since the linearization of the distance function is equivalent, after a linear change of variables, to the spherical restriction phase function.

A variation of the above methods shows that the full spherical restriction conjecture implies the full parabolic restriction conjecture. The key observation is that the restriction theorem at the critical index $p = 2n/(n+1)$ is invariant under parabolic scaling. Thus, one can scale the spherical restriction conjecture at this index until the sphere resembles a paraboloid. (Lemma 2.1 can be used to make the notion of "resembling" precise.) Of course, one does not have the restriction theorem at the endpoint index, so that in practice one only gets the localized restriction theorem $R(p, \varepsilon)$ for paraboloids for all $p < 2n/(n+1)$ and $\varepsilon > 0$. One then removes the epsilon by the techniques in Section 3. We omit the details.

Our final result is that the Kakeya and Nikodym maximal function conjectures are equivalent.

THEOREM 4.10. *If* $1 \leqslant p \leqslant n$ *and* $0 \leqslant \alpha < (n+1)/p$, *then* $K(p,\alpha) \Rightarrow N(p,\alpha) \Rightarrow K(p,2\alpha)$.

Proof. The latter implication is an analogue of Theorem 1.2 and follows immediately from the pointwise estimate

$$\delta f_\delta^*\left(\frac{x}{|x|}\right) \lesssim (f_\delta)^{**}_{C\delta^2}(x),$$

where $f_\delta(x) = f(x/\delta)$, f is a compactly supported function, and $|x| \sim 1$. This estimate reflects the geometric fact that the $1 \times \delta^2$-tubes through x that intersect the support of f_δ are essentially rescaled versions of $1 \times \delta$-tubes oriented in the direction $x/|x|$.

The former implication is an analogue of the argument in [3]. The idea is most easily expressed for the set conjecture as the observation that a Nikodym set can always be mapped onto a Kakeya set by a projective transformation. To prove the corresponding implication for the maximal operators, we first note that to prove $N(p,\alpha)$ it suffices to show the frozen estimate

$$\|f_\delta^{**}(\underline{x},0)\|_p \lesssim \delta^{(n/p)-1-\alpha}\|f\|_p,$$

where we parameterize $\mathbf{R}^n$ by $x = (\underline{x}, x_n)$. We may assume f is supported on the slab $0 < x_n \leqslant 1$. It suffices to prove the estimate for the slab $1/2 < x_n \leqslant 1$, since the condition $\alpha < (n+1)/p$ and scaling considerations ensure that the other contributions are more favourable than this main term.

The implication is now immediate from the pointwise estimate

$$f_\delta^{**}(\underline{x},0) \lesssim (f \circ \phi)^*_{C\delta}\left(\frac{(\underline{x},1)}{|(\underline{x},1)|}\right),$$

where

$$\phi(\underline{x}, x_n) = \left(\frac{\underline{x}}{x_n}, \frac{1}{x_n}\right)$$

is a projective transformation. This reflects the geometric fact that tubes through the point $(\underline{x},0)$ on the slab $\{1/2 < x_n \leqslant 1\}$ are transformed under ϕ to tubes in the direction $(\underline{x},1)/|(\underline{x},1)|$. □

REFERENCES

[1] J. BOURGAIN, *Besicovitch-type maximal operators and applications to Fourier analysis*, Geom. Funct. Anal. **1** (1991), 147–187.

[2] ———, "Some new estimates on oscillatory integrals" in *Essays on Fourier Analysis in Honor of Elias M. Stein*, Princeton Math. Ser. **42**, Princeton Univ. Press, Princeton, N.J., 1995, 83–112.

[3] A. Carbery, *Restriction implies Bochner-Riesz for paraboloids*, Math. Proc. Cambridge Philos. Soc. **111** (1992), 525–529.

[4] L. Carleson and P. Sjölin, *Oscillatory integrals and a multiplier problem for the disc*, Studia Math. **44** (1972), 287–299.

[5] M. Christ, *On almost everywhere convergence of Bochner-Riesz means in higher dimensions*, Proc. Amer. Math. Soc. **95** (1985), 16–20.

[6] ———, *Weak type endpoint bounds for Bochner-Riesz multipliers*, Rev. Mat. Iberoamericana **3** (1987), 25–31.

[7] ———, *On the regularity of inverses of singular integral operators*, Duke Math. J. **57** (1988), 459–484.

[8] ———, *Weak type* (1,1) *bounds for rough operators*, Ann. of Math. (2) **128** (1988), 19–42.

[9] M. Christ and C. D. Sogge, *The weak type L^1 convergence of eigenfunction expansions for pseudodifferential operators*, Invent. Math. **94** (1988), 421–453.

[10] A. Córdoba, *The Kakeya maximal function and spherical summation multipliers*, Amer. J. Math. **99** (1977), 1–22.

[11] C. Fefferman, *Inequalities for strongly singular convolution operators*, Acta Math. **124** (1970), 9–36.

[12] ———, *The multiplier problem for the ball*, Ann. of Math. (2) **94** (1971), 330–336.

[13] ———, *A note on spherical summation multipliers*, Israel J. Math. **15** (1973), 44–52.

[14] L. Hörmander, *Oscillatory integrals and multipliers on FL^p*, Ark. Mat. **11** (1973), 1–11.

[15] I. L. Hwang, *The L^2-boundedness of pseudodifferential operators*, Trans. Amer. Math. Soc. **302** (1987), 55–76.

[16] W. P. Minicozzi II and C. D. Sogge, *Negative results for Nikodym maximal functions and related oscillatory integrals in curved space*, Math. Res. Lett. **4** (1997), 221–237.

[17] A. Moyua, A. Vargas, and L. Vega, *Schrödinger maximal function and restriction properties of the Fourier transform*, Internat. Math. Res. Notices **1996**, 793–815.

[18] A. Seeger, *Endpoint estimates for multiplier transformations on compact manifolds*, Indiana Univ. Math. J. **40** (1991), 471–533.

[19] ———, *Endpoint inequalities for Bochner-Riesz multipliers in the plane*, Pacific J. Math. **174** (1996), 543–553.

[20] C. D. Sogge, *On the convergence of Riesz means on compact manifolds*, Ann. of Math. (2) **126** (1987), 439–447.

[21] ———, *Propagation of singularities and maximal functions in the plane*, Invent. Math. **104** (1991), 349–376.

[22] E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Math. Ser. **43**, Princeton Univ. Press, Princeton, N.J., 1993.

[23] E. M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton Math. Ser. **32**, Princeton Univ. Press, Princeton, N.J., 1971.

[24] T. Tao, *Weak-type endpoint bounds for Riesz means*, Proc. Amer. Math. Soc. **124** (1996), 2797–2805.

[25] ———, *The weak-type endpoint Bochner-Riesz conjecture and related problems*, to appear.

[26] P. A. Tomas, *A restriction theorem for the Fourier transform*, Bull. Amer. Math. Soc. **81** (1975), 477–478.

[27] T. H. Wolff, *Recent work on sharp estimates in second-order elliptic unique continuation problems*, J. Geom. Anal. **3** (1993), 621–650.

[28] ———, *An improved bound for Kakeya type maximal functions*, Rev. Mat. Iberoamericana **11** (1995), 651–674.

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