

INTERTWINING OPERATOR ALGEBRAS AND VERTEX
TENSOR CATEGORIES FOR AFFINE LIE ALGEBRAS

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0. Introduction. The category of finite direct sums of standard (integrable highest weight) modules of a fixed positive integral level k for an affine Lie algebra $\hat{\mathfrak{g}}$ is particularly important from the viewpoint of conformal field theory and related mathematics. Here we call this the category *generated by* the standard $\hat{\mathfrak{g}}$ -modules of level k . A central theme is a braided tensor category structure (in the sense of Joyal and Street [JS]) on this category, a structure explicitly discovered by Moore and Seiberg [MS] in their important study of conformal field theories. In [MS], Moore and Seiberg constructed this structure based on the assumption that there exists a suitable operator product expansion for chiral vertex operators; this is essentially equivalent to assuming the associativity of intertwining operators, in the language of vertex operator algebra theory. Actually, Belavin, Polyakov, and Zamolodchikov [BPZ] had already formalized the relation between the operator product expansion and representation theory in the context of conformal field theory, especially for the Virasoro algebra, and Knizhnik and Zamolodchikov [KZ] had established the relation between conformal field theory and the representation theory of affine Lie algebras.

In [KL1]–[KL5], Kazhdan and Lusztig achieved a breakthrough by indeed constructing a natural braided tensor category structure, with the additional property of rigidity, on a certain category of $\hat{\mathfrak{g}}$ -modules of level k , when k is sufficiently negative, or more generally, when k is in a certain large subset of \mathbb{C} excluding the positive integers, and particularly, proving that this braided tensor category is equivalent to a tensor category of modules for a quantum group constructed from the same finite-dimensional Lie algebra. The method used by Kazhdan and Lusztig, especially in their construction of the associativity isomorphisms, is algebro-geometric and is closely related to the algebro-geometric formulation and study of conformal-field-theoretic structures in the influential works of Tsuchiya, Ueno, and Yamada [TUY], Drinfeld [Dr], and Beilinson, Feigin, and Mazur [BFM]. (The work [BFM] discusses the case of minimal models for the Virasoro algebra, and Beilinson informs us that a similar argument works for the case of the category generated by the standard $\hat{\mathfrak{g}}$ -modules.)

In the important work [F1] and [F2], Finkelberg proved that the braided tensor

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category at positive level is tensor equivalent to a certain “subquotient” of the braided tensor category constructed by Kazhdan and Lusztig at negative level, and is thus equivalent to a certain “subquotient” braided tensor category of quantum group representations. (Cf. also [Va].) We have been informed by Finkelberg that the arguments in his paper [F2] can in fact be reinterpreted to actually give a proof of the coherence relations for the braided tensor category structure on the category generated by the standard $\hat{\mathfrak{g}}$ -modules of positive integral level k ; this reasoning uses Kazhdan and Lusztig’s result mentioned above, constructing a rigid braided tensor category structure at negative level, and it also uses Deligne’s reformulation [De] of the notion of balanced braided tensor category. (Finkelberg also informs us that his arguments do not establish the coherence in a few cases, namely, those of $\mathfrak{g} = E_6, E_7, E_8$, for $k = 1$, or $\mathfrak{g} = E_8$, for $k = 2$, because of the corresponding restrictions at negative level in [KL1]–[KL5].)

In this paper, we prove the associativity of intertwining operators (the existence of operator product expansion for chiral vertex operators) for the vertex operator algebras associated to affine Lie algebras at positive level, and we thereby directly construct the braided tensor category structure on the category generated by the standard $\hat{\mathfrak{g}}$ -modules of positive integral level, as an application of the general tensor product theory for representations of a vertex operator algebra developed in our papers [HL1]–[HL6] and [H1]–[H2]. We also prove more general results, and in fact the methods in our general theory are very different from those in the works [MS], [KL1]–[KL5], [TUY], [Dr], [BFM], [F1], and [F2] mentioned above. We hope that the present work helps illustrate the viewpoint that vertex operator algebra theory is the appropriate mathematical foundation for understanding the range of conformal-field-theoretic structures, including the associativity of intertwining operators and the braided tensor category structure.

What the present paper specifically does is to use the works of Knizhnik and Zamolodchikov [KZ], Tsuchiya and Kanie [TK], Frenkel, Huang, and Lepowsky [FHL], Frenkel and Zhu [FZ], Dong, Li, and Mason [DLM], Dong and Mason [DM], Dong, Mason, and Zhu [DMZ], and Li [L1] and [L2] to verify the technical conditions needed to apply our tensor product theory in the particular case of the conformal field theories associated with affine Lie algebras (or Wess-Zumino-Novikov-Witten (WZNW) models) and related models in conformal field theory; the main part of our construction is already contained in [HL1]–[HL6] and especially [H1]–[H2]. We would like to emphasize that the fundamental ideas in [KZ], [TK], and [FZ] play essential roles in the present paper, as does [FHL].

Our general theory in [HL1]–[HL6] and [H1]–[H2] (in the generality of suitable vertex operator algebras) was initially motivated by Kazhdan and Lusztig’s major series of papers [KL1]–[KL5]. The category studied by Kazhdan and Lusztig is significantly different from the category generated by the standard $\hat{\mathfrak{g}}$ -modules of level k . It is larger; it consists of the modules of level k of finite length whose composition factors are irreducible highest weight modules corresponding to weights that

are dominant integral in the direction of \mathfrak{g} . For k in the range studied by Kazhdan and Lusztig, there are no nonzero standard $\hat{\mathfrak{g}}$ -modules of level k , and for $k \in \mathbb{Z}_+$, there is no braided tensor category structure on the category considered by Kazhdan and Lusztig (as is pointed out in their work). Correspondingly, our methods are very different from Kazhdan and Lusztig's, with an important exception: The *definition* of the tensor product operation (for general vertex operator algebras) in our work (see [HL1]–[HL4]) is analogous to and was inspired by that in [KL1]–[KL5]. However, our approach is based on a certain determination of the tensor product module (see [HL1]–[HL5]). Moreover, the *construction of the associativity isomorphisms* (see [H1]) uses a completely different method from the one in [KL1]–[KL5]; a large amount of the information needed in the proof of the necessary coherence properties (i.e., the pentagon and hexagon properties) is already encoded in the associativity isomorphisms.

The associativity of intertwining operators for the vertex operator algebra associated to an affine Lie algebra at positive level proved in this paper implies that the direct sum of all inequivalent irreducible modules for the vertex operator algebra forms an intertwining operator algebra, in the sense of [H3], [H5], and [H6]. In fact, we are also able to obtain what we have called a “vertex tensor category” structure (see [HL2] and [HL6]), which is much richer than a braided tensor category structure; it is defined in terms of certain moduli spaces of genus-zero Riemann surfaces with punctures and local coordinates, equipped with the sewing operation (see [H4]). Vertex tensor category structure automatically yields braided tensor category structure by the process of ignoring the conformal-geometric information and retaining only the topological information. We further obtain the corresponding results for certain vertex operator algebras slightly more general than those associated with the WZNW models. As we said above, we do this here by verifying the technical conditions needed to apply our general theory.

It was proved in [H1] that the existence of the associativity isomorphisms is equivalent to the associativity of intertwining operators, in the language of vertex operator algebra theory. In their fundamental paper [TK], Tsuchiya and Kanie studied (in the special case $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$) correlation functions constructed from products of several intertwining operators, and they used these correlation functions to construct and study braid group representations. Their main tool was the classic Knizhnik-Zamolodchikov (KZ) differential equations [KZ]. In the present paper, we use this same method to verify a basic technical condition, namely, the convergence and extension property, a key condition needed for the application of the main associativity theorem in [H1]. However, the definition of intertwining operator (or chiral vertex operator) in [TK] is different from the one in [MS] or the one in [FHL]. The definition in [TK] (as in many other works) is restricted to primary fields (i.e., intertwining operators in the sense of [FHL] associated with the lowest conformal-weight vectors of a module), and the iterated action of $\hat{\mathfrak{g}}$ on these operators is also fundamentally used. This restricted definition of intertwining operator is intimately related to the

prevalent notion of correlation function based on certain Lie algebra coinvariants (as in [KL1]–[KL5], for example). Our notion of correlation function is based instead on products of intertwining operators in the sense of [FHL].

The “nuclear democracy theorem” in [TK] can be reinterpreted as asserting the equivalence of these two different notions of intertwining operator in the case $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$, but iterates of intertwining operators themselves, essential to the formulation and proof of the associativity of intertwining operators, were not studied in [TK]. The proof of the results in [TK] necessary for the construction of the braid group representations used the KZ equations as well as certain algebraic constraints based on null-vector conditions.

The notion of intertwining operator in [FHL] (which we use in the present paper) is the notion that is natural from the general perspective of vertex operator algebra theory. (The starting point is the Jacobi identity for vertex operator algebras [FLM]; cf. [B].) In fact, the theory in [HL1]–[HL6] and [H1]–[H2], based on this notion, automatically incorporates the corresponding algebraic constraints and the subtlety of “nuclear democracy” at every stage, and the theory works in considerable generality. This notion has led us to consider iterates of intertwining operators, to formulate the associativity of intertwining operators (in terms of such iterates), and to formulate a “compatibility condition,” which was used to construct our tensor products in [HL3] and [HL4]. On the other hand, it was shown by Li [L1] and [L3] that the nuclear democracy theorem generalizes to any \mathfrak{g} , and thus for any \mathfrak{g} , the notions of intertwining operator in the sense of [TK] and in the sense of [FHL] are indeed equivalent.

In [H1], the associativity of intertwining operators was proved using our theory of tensor products of modules for a vertex operator algebra developed in [HL3]–[HL5], when the vertex operator algebra is rational in the sense of [HL3] and satisfies certain additional technical conditions. Combined with the results of Moore and Seiberg [MS], the result of [H1] served to construct a natural braided tensor category structure on the category of modules for such a vertex operator algebra. We also showed in [H2] and [HL6] that when the vertex operator algebra contains a rational vertex operator subalgebra (but may itself not be rational) and satisfies the additional technical conditions mentioned above, we still get a braided tensor category structure such that the tensor product bifunctor is the bifunctor $\boxtimes_{P(1)}$ constructed in [HL5], and in fact, we get a vertex tensor category structure. In [H2], it was verified that for a vertex operator algebra containing a vertex operator subalgebra isomorphic to a tensor product of minimal Virasoro vertex operator algebras, the technical conditions for the applicability of the tensor product theory are satisfied. Thus the category of modules for such a vertex operator algebra has a natural vertex tensor category structure and, in particular, a natural braided tensor category structure.

In the present paper, we work in the following generality, somewhat greater than the generality of vertex operator algebras associated with $\hat{\mathfrak{g}}$ and positive integral level k : We verify that for a vertex operator algebra containing a vertex operator subalgebra isomorphic to a tensor product of vertex operator algebras associated with WZNW

models, the technical conditions for the applicability of our tensor product theory are satisfied. (Recall from [FHL] the notions of tensor product and subalgebra for vertex operator algebras; in particular, a subalgebra is required to have the same Virasoro element as the large algebra.) We thereby complete the construction of the desired vertex tensor category structure and, in particular, braided tensor category structure, on the category of modules for such a vertex operator algebra. In the special case of a vertex operator algebra associated with a WZNW model, we conclude in particular that the category generated by the standard $\hat{\mathfrak{g}}$ -modules of positive integral level k is indeed a braided tensor category. See Section 3 for the main results.

The main tool that we use in this paper to verify our conditions is the classical system of KZ equations. Since we work in the framework of the theory of vertex operator algebras, we include an exposition in the language of vertex operator algebras of the fundamental theorem in [KZ] that products of intertwining operators for a vertex operator algebra associated with an affine Lie algebra satisfy the KZ equations. Our treatment of this result is nothing but an adaptation of the original argument in [KZ], except that we use the language of formal variables, in the spirit of the present theory; cf. the treatment of the KZ equations in [TK]. As in [KZ] and [TK], the theorem that products of intertwining operators satisfy the KZ equations must be understood as an assertion about formal series and not about functions of complex variables, since one does not yet know at this stage that these products are convergent; in fact, the KZ equations are used to *prove* this convergence of products of intertwining operators.

We assume that the reader is familiar with the statements of the main results of [HL1]–[HL6] and [H1]–[H2]. We use the basics of formal calculus and vertex operator algebra theory as presented in [FHL], for example. We shall assume in this paper that in the definition of a module for a vertex operator algebra, the grading is by \mathbb{C} , not \mathbb{Q} (see [FHL, Remark 4.1.2]). Also recall that a module, by definition, has two grading-restriction properties—finite-dimensionality of the weight spaces and lower truncation of the grading, and in this paper, when we assert that a structure is a module, these restrictions are often the main issue.

In Section 1, we briefly review vertex operator algebras associated with affine Lie algebras and their representations. The results in this section are due to Frenkel and Zhu [FZ] (see also [DL] and [L2]). In Section 2, we present our exposition that products of intertwining operators for a vertex operator algebra associated with an affine Lie algebra satisfy the KZ equations. In Section 3, we use the KZ equations and results obtained in [HL1]–[HL6] and [H1]–[H2], combined with results in [DLM], [DM], [DMZ], [FHL], and [L1], to prove our main results.

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1. Vertex operator algebras associated with affine Lie algebras. Let \mathfrak{g} be a Lie algebra over \mathbb{C} equipped with an invariant symmetric bilinear form (\cdot, \cdot) . The *affine Lie algebra* $\hat{\mathfrak{g}}$ associated with \mathfrak{g} and (\cdot, \cdot) is the vector space $\mathfrak{g} \otimes [t, t^{-1}] \oplus \mathbb{C}\mathbf{k}$ equipped with the bracket operation defined by

$$\begin{aligned} [a \otimes t^m, b \otimes t^n] &= [a, b] \otimes t^{m+n} + (a, b)m\delta_{m+n,0}\mathbf{k}, \\ [a \otimes t^m, \mathbf{k}] &= 0, \end{aligned}$$

for $a, b \in \mathfrak{g}$ and $m, n \in \mathbb{Z}$. It is \mathbb{Z} -graded in a natural way. Consider the subalgebras

$$\hat{\mathfrak{g}}_{\pm} = \mathfrak{g} \otimes t^{\pm 1} \mathbb{C}[t^{\pm 1}] \quad \text{and} \quad \mathfrak{g} = \mathfrak{g} \otimes \mathbb{C} \quad \text{of} \quad \hat{\mathfrak{g}}$$

and the vector space decomposition

$$\hat{\mathfrak{g}} = \hat{\mathfrak{g}}_- \oplus \mathfrak{g} \oplus \mathbb{C}\mathbf{k} \oplus \hat{\mathfrak{g}}_+.$$

Let M be a \mathfrak{g} -module, viewed as homogeneously graded of fixed-degree $l \in \mathbb{C}$, and let $k \in \mathbb{C}$. Let $\hat{\mathfrak{g}}_+$ act trivially on M and \mathbf{k} as the scalar multiplication operator k . Then M becomes a $\mathfrak{g} \oplus \mathbb{C}\mathbf{k} \oplus \hat{\mathfrak{g}}_+$ -module, and we have the \mathbb{C} -graded induced $\hat{\mathfrak{g}}$ -module

$$\hat{M}_k = U(\hat{\mathfrak{g}}) \otimes_{U(\mathfrak{g} \oplus \mathbb{C}\mathbf{k} \oplus \hat{\mathfrak{g}}_+)} M,$$

which contains a canonical copy of M , of degree l .

Now we assume that \mathfrak{g} is finite-dimensional and simple. Let \mathfrak{h} be a Cartan subalgebra, Δ the root system of $(\mathfrak{g}, \mathfrak{h})$, $\Delta_+ \subset \Delta$ a fixed set of positive roots, θ the highest root, h^\vee the dual Coxeter number of \mathfrak{g} , and $(\cdot, \cdot) : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C}$ the Cartan-Killing form, normalized by the condition $(\theta, \theta) = 2$, where (\cdot, \cdot) is transported canonically to $\mathfrak{h}^* \times \mathfrak{h}^*$. For a fixed $\lambda \in \mathfrak{h}^*$, let $L(\lambda)$ be the irreducible highest weight \mathfrak{g} -module with highest weight λ , assigned a homogeneous degree in \mathbb{C} . We shall use the notation $M(k, \lambda)$ to denote the graded $\hat{\mathfrak{g}}$ -module $\widehat{L(\lambda)}_k$ (the degree of $L(\lambda)$ being suppressed in the notation). Let $J(k, \lambda)$ be the maximal proper graded submodule of $M(k, \lambda)$ and $L(k, \lambda) = M(k, \lambda)/J(k, \lambda)$. Then $L(k, \lambda)$ is the unique irreducible graded $\hat{\mathfrak{g}}$ -module such that \mathbf{k} acts as k and the space of all elements annihilated by $\hat{\mathfrak{g}}_+$ is isomorphic to the \mathfrak{g} -module $L(\lambda)$. Later, we shall often use the same notation to denote elements of $M(k, \lambda)$ and of $L(k, \lambda)$.

In the special case $\lambda = 0$, $L(0)$ is the 1-dimensional trivial \mathfrak{g} -module (equipped with a homogeneous degree in \mathbb{C}) and can be identified with \mathbb{C} . Consequently, as a vector space and $\hat{\mathfrak{g}}_-$ -module, $M(k, 0)$ is naturally isomorphic to the universal enveloping algebra $U(\hat{\mathfrak{g}}_-)$. We define a vertex operator map

$$\begin{aligned} Y(\cdot, x) : M(k, 0) &\rightarrow (\text{End } M(k, 0))[[x, x^{-1}]] \\ v &\mapsto Y(v, x) = \sum_{n \in \mathbb{Z}} v_n x^{-n-1}, \end{aligned}$$

as follows: Identifying $M(k, 0)$ with $U(\hat{\mathfrak{g}}_-)$, we note that $M(k, 0)$ is spanned by the elements of the form $a_1(-n_1) \cdots a_m(-n_m)1$, where $a_1, \dots, a_m \in \mathfrak{g}$ and $n_1, \dots, n_m \in \mathbb{Z}_+$, with $a(-n)$ denoting the representation image of $a \otimes t^{-n}$ for $a \in \mathfrak{g}$ and $n \in \mathbb{Z}$

(we shall use similar notation for other $\hat{\mathfrak{g}}$ -modules below). We use recursion on m to define the vertex operator map. For $1 \in M(k, 0)$, we define $Y(1, x)$ to be the identity operator on $M(k, 0)$, and for $a \in \mathfrak{g}$, we define

$$Y(a(-1)1, x) = \sum_{n \in \mathbb{Z}} a(n)x^{-n-1}.$$

Assume that the map has been defined for $a_1(-n_1) \cdots a_m(-n_m)1$. Vertex operators for elements of the form $a_0(-n_0)a_1(-n_1) \cdots a_m(-n_m)1$ are defined using the residue in x_1 of the Jacobi identity for vertex operator algebras (cf. (2.1) below) as follows:

$$\begin{aligned} & Y(a_0(-n_0)a_1(-n_1) \cdots a_m(-n_m)1, x) \\ &= \text{Res}_{x_1} (x_1 - x)^{-n_0} Y(a_0(-1)1, x_1) Y(a_1(-n_1) \cdots a_m(-n_m)1, x) \\ & \quad - \text{Res}_{x_1} (-x + x_1)^{-n_0} Y(a_1(-n_1) \cdots a_m(-n_m)1, x) Y(a_0(-1)1, x_1). \end{aligned} \quad (1.1)$$

(Here and below, binomial expressions are understood to be expanded in nonnegative powers of the second variable.) The vacuum vector for $M(k, 0)$ is $\mathbf{1} = 1$. In the case that $k \neq -h^\vee$, $M(k, 0)$ also has a Virasoro element

$$\omega = \frac{1}{2(k + h^\vee)} \sum_{i=1}^{\dim \mathfrak{g}} g^i (-1)^2 \mathbf{1}, \quad (1.2)$$

where $\{g^i\}_{i=1, \dots, \dim \mathfrak{g}}$ is an arbitrary orthonormal basis of \mathfrak{g} with respect to the form (\cdot, \cdot) . With respect to the associated grading, $\mathbf{1}$ has weight zero; the (conformal) weight of a homogeneous element is the negative of its degree. In [FZ], the following result is proved (among other things).

THEOREM 1.1. *If $k \neq -h^\vee$, the quadruple $(M(k, 0), Y, \mathbf{1}, \omega)$ defined above is a vertex operator algebra; in particular, $Y(\cdot, x)$ is well defined.*

Since $J(k, 0)$ is a $\hat{\mathfrak{g}}$ -submodule of $M(k, 0)$, the vertex operator map for $M(k, 0)$ induces a vertex operator map for $L(k, 0)$, which we still denote Y . We continue to denote the $J(k, 0)$ -cosets of $\mathbf{1}$ and ω in $M(k, 0)$ by $\mathbf{1}$ and ω . The following result is an immediate consequence of Theorem 1.1.

COROLLARY 1.2 [FZ]. *If $k \neq -h^\vee$, the quadruple $(L(k, 0), Y, \mathbf{1}, \omega)$ is a (nonzero) vertex operator algebra.*

We now discuss modules for these vertex operator algebras. For any $k \in \mathbb{C}$ and $\lambda \in \mathfrak{h}^*$, we define a vertex operator map

$$Y(\cdot, x) : M(k, 0) \rightarrow (\text{End } M(k, \lambda))[[x, x^{-1}]]$$

using recursion, just as in (1.1). As in Theorem 1.1, this is well defined. The induced map

$$Y(\cdot, x) : L(k, 0) \rightarrow (\text{End } L(k, \lambda))[[x, x^{-1}]]$$

is well defined under the conditions given in the following result, also proved in [FZ];

see also [L2].

THEOREM 1.3. *For any $k \in \mathbb{C}$ such that $k \neq -h^\vee$, and any $\lambda \in \mathfrak{h}^*$ with λ dominant integral (i.e., with $\dim L(\lambda) < \infty$), the pair $(M(k, \lambda), Y)$ is a module for the vertex operator algebra $M(k, 0)$. In the case $k = 0, 1, 2, \dots$, $L(k, 0)$ is a rational vertex operator algebra and*

$$\{L(k, \lambda) \mid \lambda \in \mathfrak{h}^* \text{ is dominant integral such that } (\lambda, \theta) \leq k\}$$

is the set of all irreducible $L(k, 0)$ -modules up to equivalence.

Remark 1.4. In the present paper, *rationality* for a vertex operator algebra refers to the definition in [HL3]: the number of inequivalent irreducible modules is finite, every module is completely reducible, and the fusion rules (the dimensions of the spaces of intertwining operators among triples of modules) are finite. The notion of rationality used in [FZ] was a different one, but the rationality of $L(k, 0)$ in our sense in fact holds.

Remark 1.5. When $k \in \mathbb{N}$, the vertex operator algebras $L(k, 0)$ and their irreducible modules can also be constructed explicitly using Fock spaces and tensor products. See [B], [FLM], and [DL, Chapter 13].

2. Products of intertwining operators and the KZ equations. Here we give our exposition of the following: When $k \in \mathbb{N}$, products of intertwining operators for the vertex operator algebra $L(k, 0)$ satisfy a system of differential equations with regular singular points, called the *Knizhnik-Zamolodchikov (KZ) equations*, which were first derived in [KZ]. The reason for our supplying this exposition was explained in the introduction. The discussions and results in this section also hold for the vertex operator algebras $M(k, 0)$, for any $k \in \mathbb{C}$ such that $k \neq -h^\vee$.

In this and the next section, we need a simple general principle for vertex operators and intertwining operators (see (2.3) below; cf. [DL, (13.24)–(13.26)]). We shall use the basics of formal calculus as expressed in [FHL, Section 2.1], in particular, the binomial expansion convention.

Let V be a vertex operator algebra, let W_1, W_2 , and W_3 be V -modules, and let \mathcal{Y} be an intertwining operator of type $\binom{W_3}{W_1 W_2}$. Then, from the Jacobi identity defining \mathcal{Y} (see [FHL, (5.4.4)]), we obtain, by taking Res_{x_1} ,

$$\begin{aligned} \mathcal{Y}(Y(u, x_0)w, x_2) &= \left(\text{Res}_{x_1} x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) Y(u, x_1) \right) \mathcal{Y}(w, x_2) \\ &\quad - \mathcal{Y}(w, x_2) \left(\text{Res}_{x_1} x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) Y(u, x_1) \right), \end{aligned} \quad (2.1)$$

for $u \in V$ and $w \in W_1$ (cf. (1.1)).

We write

$$Y(u, x) = Y^+(u, x) + Y^-(u, x), \quad (2.2)$$

where $Y^+(u, x) = \sum_{n < 0} u_n x^{-n-1}$ and $Y^-(u, x) = \sum_{n \geq 0} u_n x^{-n-1}$ are the regular and singular parts of $Y(u, x)$, respectively. Equating the parts of both sides of (2.1), which are constant in x_0 , we obtain

$$\begin{aligned}
 & \mathfrak{Y}(u_{-1}w, x_2) \\
 &= \left(\text{Res}_{x_1} (x_1 - x_2)^{-1} Y(u, x_1) \right) \mathfrak{Y}(w, x_2) + \mathfrak{Y}(w, x_2) \left(\text{Res}_{x_1} (x_2 - x_1)^{-1} Y(u, x_1) \right) \\
 &= \left(\text{Res}_{x_1} x_1^{-1} \delta \left(\frac{x_2}{x_1} \right) Y^+(u, x_1) \right) \mathfrak{Y}(w, x_2) + \mathfrak{Y}(w, x_2) \left(\text{Res}_{x_1} x_1^{-1} \delta \left(\frac{x_2}{x_1} \right) Y^-(u, x_1) \right) \\
 &= \left(\text{Res}_{x_1} x_1^{-1} \delta \left(\frac{x_2}{x_1} \right) Y^+(u, x_2) \right) \mathfrak{Y}(w, x_2) + \mathfrak{Y}(w, x_2) \left(\text{Res}_{x_1} x_1^{-1} \delta \left(\frac{x_2}{x_1} \right) Y^-(u, x_2) \right) \\
 &= Y^+(u, x_2) \mathfrak{Y}(w, x_2) + \mathfrak{Y}(w, x_2) Y^-(u, x_2) \\
 &= \circ Y(u, x_2) \mathfrak{Y}(w, x_2) \circ, \tag{2.3}
 \end{aligned}$$

where we define the ‘‘normal ordering’’ $\circ \cdot \circ$ by

$$\circ u_m w_n \circ = \begin{cases} u_m w_n, & \text{if } m < 0, \\ w_n u_m, & \text{if } m \geq 0. \end{cases} \tag{2.4}$$

Note that this particular normal-ordering operation is not, in general, commutative; moreover, the expansion $\mathfrak{Y}(w, x) = \sum_{n \in \mathbb{C}} w_n x^{-n-1}$ involves nonintegral subscripts n in general.

Now we take V to be one of the vertex operator algebras $L(k, 0)$, for $k \in \mathbb{N}$, or $M(k, 0)$, for $k \in \mathbb{C}$ and $k \neq -h^\vee$. Let W be any V -module. For any $g \in \mathfrak{g}$, we denote $Y(g(-1)\mathbf{1}, x)$, $Y^+(g(-1)\mathbf{1}, x)$, and $Y^-(g(-1)\mathbf{1}, x)$, acting on W , by $g(x)$, $g^+(x)$, and $g^-(x)$, respectively.

From (1.2) and (2.3) (note that $g^i(-1) = (g^i(-1)\mathbf{1})_{-1}$), we obtain

$$Y(\omega, x) = \frac{1}{2(k+h^\vee)} \sum_{i=1}^{\dim \mathfrak{g}} : g^i(x)^2 :, \tag{2.5}$$

$$L(n) = \frac{1}{2(k+h^\vee)} \sum_{i=1}^{\dim \mathfrak{g}} \sum_{j \in \mathbb{Z}} : g^i(j) g^i(n-j) :, \tag{2.6}$$

for $n \in \mathbb{Z}$, where, for any $g, g' \in \mathfrak{g}$,

$$: g(r) g'(s) := \begin{cases} g(r) g'(s), & \text{if } r < s, \\ \frac{1}{2} (g(r) g'(s) + g'(s) g(r)), & \text{if } r = s, \\ g'(s) g(r), & \text{if } r > s \end{cases}$$

(cf. [DL, Proposition 13.4 and (13.35)]; in the present situation, the two normal orderings coincide). In particular,

$$L(-1) = \frac{1}{2(k+h^\vee)} \sum_{i=1}^{\dim \mathfrak{g}} \left(\sum_{j<0} g^i(j)g^i(-j-1) + \sum_{j\geq 0} g^i(-j-1)g^i(j) \right).$$

If $w \in W$ is such that $g(n)w = 0$ for $n > 0$ and $g \in \mathfrak{g}$, then

$$L(-1)w = \frac{1}{k+h^\vee} \sum_{i=1}^{\dim \mathfrak{g}} g^i(-1)g^i(0)w = \frac{1}{k+h^\vee} \sum_{i=1}^{\dim \mathfrak{g}} g^i(-1)g^i w. \quad (2.7)$$

Let $n \geq 1$, let W_0, \dots, W_{n-1} be V -modules, and let $W_n = V$. We shall consider a product of intertwining operators of the form

$$\mathfrak{y}_1(w_{(1)}, x_1) \cdots \mathfrak{y}_n(w_{(n)}, x_n), \quad (2.8)$$

for $n \geq 1$. Here $w_{(l)} \in L(\lambda_l)$, where $\lambda_l \in \mathfrak{h}^*$ is dominant integral. Recall Theorem 1.3. In the case in which $V = L(k, 0)$ ($k \in \mathbb{N}$), we assume that $(\lambda_l, \theta) \leq k$ and view $L(\lambda_l)$ as the lowest conformal-weight space of the V -module $L(k, \lambda_l)$, and in the case $V = M(k, 0)$ ($k \neq -h^\vee$), we analogously take $L(\lambda_l)$ to be the lowest conformal-weight space of the V -module $M(k, \lambda_l)$. The intertwining operator \mathfrak{y}_l is of type $\binom{\tilde{W}_{l-1}}{L(k, \lambda_l)W_l}$ in the former case and of type $\binom{W_{l-1}}{M(k, \lambda_l)W_l}$ in the latter case.

By the $L(-1)$ -derivative property in the definition of intertwining operator and (2.7),

$$\begin{aligned} (k+h^\vee) \frac{d}{dx_l} \mathfrak{y}_l(w_{(l)}, x_l) &= (k+h^\vee) \mathfrak{y}_l(L(-1)w_{(l)}, x_l) \\ &= \sum_{i=1}^{\dim \mathfrak{g}} \circ g^i(x_l) \mathfrak{y}_l(g^i w_{(l)}, x_l) \circ. \end{aligned} \quad (2.9)$$

For any $g \in \mathfrak{g}$ and $w_{(p)} \in L(k, \lambda_p)$ with $p \neq l$,

$$\begin{aligned} [g(x_l), \mathfrak{y}_p(w_{(p)}, x_p)] &= \text{Res}_{x_0} x_p^{-1} \delta \left(\frac{x_l - x_0}{x_p} \right) \mathfrak{y}_p(g(x_0)w_{(p)}, x_p) \\ &= \text{Res}_{x_0} x_p^{-1} \delta \left(\frac{x_l - x_0}{x_p} \right) \mathfrak{y}_p \left(\sum_{n \leq 0} g(n)x_0^{-n-1} w_{(p)}, x_p \right) \\ &= x_p^{-1} \delta \left(\frac{x_l}{x_p} \right) \mathfrak{y}_p(gw_{(p)}, x_p), \end{aligned}$$

and so

$$[g^-(x_l), \mathfrak{y}_p(w_{(p)}, x_p)] = (x_l - x_p)^{-1} \mathfrak{y}_p(gw_{(p)}, x_p), \quad (2.10)$$

$$[g^+(x_l), \mathfrak{y}_p(w_{(p)}, x_p)] = -(-x_p + x_l)^{-1} \mathfrak{y}_p(gw_{(p)}, x_p), \quad (2.11)$$

where we continue to use the binomial expansion convention.

We introduce the following notation: For commuting formal variables x_1, \dots, x_n , let

$$: (x_l - x_p)^{-1} := \begin{cases} (x_l - x_p)^{-1}, & \text{if } l < p, \\ (-x_p + x_l)^{-1}, & \text{if } p < l. \end{cases}$$

Consider the contragredient V -module W'_0 (see [FHL, Sections 5.2 and 5.3]), and let $w'_{(0)} \in W'_0$ be a lowest conformal-weight vector. By (2.9)–(2.11), and since

$$\begin{aligned} (g^i)^-(x_l)\mathbf{1} &= 0, \\ \langle w'_{(0)}, (g^i)^+(x_l)u \rangle &= 0 \end{aligned} \quad (2.12)$$

for any $u \in V$, we have (for fixed l)

$$\begin{aligned} & (k + h^\vee) \frac{d}{dx_l} \langle w'_{(0)}, \mathfrak{y}_1(w_{(1)}, x_1) \cdots \mathfrak{y}_n(w_{(n)}, x_n) \mathbf{1} \rangle \\ &= \sum_{p \neq l} : (x_l - x_p)^{-1} : \sum_{i=1}^{\dim \mathfrak{g}} \langle w'_{(0)}, \mathfrak{y}_1(w_{(1)}, x_1) \cdots \mathfrak{y}_l(g^i w_{(l)}, x_l) \cdots \\ & \quad \mathfrak{y}_p(g^i w_{(p)}, x_p) \cdots \mathfrak{y}_n(w_{(n)}, x_n) \mathbf{1} \rangle. \end{aligned} \quad (2.13)$$

We define operators Ω_{lp} , for $1 \leq l, p \leq n$ with $l \neq p$, on

$$(L(\lambda_1) \otimes \cdots \otimes L(\lambda_n))^*,$$

as follows:

$$(\Omega_{lp} f)(w_{(1)} \otimes \cdots \otimes w_{(n)}) = \sum_{i=1}^{\dim \mathfrak{g}} f(w_{(1)} \otimes \cdots \otimes g^i w_{(l)} \otimes \cdots \otimes g^i w_{(p)} \otimes \cdots \otimes w_{(n)}).$$

Note that $\Omega_{lp} = \Omega_{pl}$. We extend Ω_{lp} naturally to an operator on the vector space

$$(L(\lambda_1) \otimes \cdots \otimes L(\lambda_n))^* \{x_1, \dots, x_n\}$$

of formal series with arbitrary powers of the variables and with coefficients in $(L(\lambda_1) \otimes \cdots \otimes L(\lambda_n))^*$. In terms of these operators, (2.13) becomes the following theorem.

THEOREM 2.1 (Knizhnik and Zamolodchikov). *Let $V = L(k, 0)$, for $k \in \mathbb{N}$, or $V = M(k, 0)$, for $k \in \mathbb{C}$ and $k \neq -h^\vee$. Let $n \geq 1$, let W_0, \dots, W_{n-1} be V -modules, and let $W_n = V$. Let $\mathfrak{y}_1, \dots, \mathfrak{y}_n$ be intertwining operators, $w_{(l)} \in L(\lambda_l)$ and $w'_{(0)} \in W'_0$, as described above (see (2.8) and (2.12)), and define*

$$\phi_{\mathfrak{y}_1, \dots, \mathfrak{y}_n} \in (L(\lambda_1) \otimes \cdots \otimes L(\lambda_n))^* \{x_1, \dots, x_n\}$$

by

$$\phi_{\mathfrak{y}_1, \dots, \mathfrak{y}_n}(w_{(1)} \otimes \dots \otimes w_{(n)}) = \left\langle w'_{(0)}, \mathfrak{y}_1(w_{(1)}, x_1) \cdots \mathfrak{y}_n(w_{(n)}, x_n) \mathbf{1} \right\rangle.$$

Then for $l = 1, \dots, n$,

$$(k + h^\vee) \frac{\partial}{\partial x_l} \phi_{\mathfrak{y}_1, \dots, \mathfrak{y}_n} = \sum_{p \neq l} : (x_l - x_p)^{-1} : \Omega_{lp} \phi_{\mathfrak{y}_1, \dots, \mathfrak{y}_n}. \quad (2.14)$$

The equations (2.14) are the *KZ equations*. They are easily seen to be consistent. Theorem 2.1 gives the following corollary.

COROLLARY 2.2. *Let $k \in \mathbb{N}$; let $\lambda_l \in \mathfrak{h}^*$, for $l = 0, 1, 2, 3$, be dominant integral weights satisfying $(\lambda_l, \theta) \leq k$; let W be an $L(k, 0)$ -module; and let \mathfrak{y}_1 and \mathfrak{y}_2 be intertwining operators of types $\binom{L(k, \lambda_0)}{L(k, \lambda_1)W}$ and $\binom{W}{L(k, \lambda_2)L(k, \lambda_3)}$, respectively. Define*

$$\psi_{\mathfrak{y}_1, \mathfrak{y}_2} \in (L(\lambda_0)^* \otimes L(\lambda_1) \otimes L(\lambda_2) \otimes L(\lambda_3))^* \{x_1, x_2\}$$

by

$$\psi_{\mathfrak{y}_1, \mathfrak{y}_2}(w'_{(0)} \otimes w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) = \left\langle w'_{(0)}, \mathfrak{y}_1(w_{(1)}, x_1) \mathfrak{y}_2(w_{(2)}, x_2) w_{(3)} \right\rangle,$$

for $w'_{(0)} \in L(\lambda_0)^* \subset L(k, \lambda_0)'$ and $w_{(l)} \in L(\lambda_l) \subset L(k, \lambda_l)$, with $l = 1, 2, 3$. Then

$$(k + h^\vee) \frac{\partial}{\partial x_1} \psi_{\mathfrak{y}_1, \mathfrak{y}_2} = \left(\frac{\Omega_{13}}{x_1} + \frac{\Omega_{12}}{x_1 - x_2} \right) \psi_{\mathfrak{y}_1, \mathfrak{y}_2}, \quad (2.15)$$

$$(k + h^\vee) \frac{\partial}{\partial x_2} \psi_{\mathfrak{y}_1, \mathfrak{y}_2} = \left(\frac{\Omega_{23}}{x_2} + \frac{\Omega_{12}}{-x_1 + x_2} \right) \psi_{\mathfrak{y}_1, \mathfrak{y}_2}. \quad (2.16)$$

3. The category of standard modules of a fixed level for an affine Lie algebra.

In this section, we show that for the category of modules for a vertex operator algebra containing a subalgebra isomorphic to a tensor product of vertex operator algebras of the form $L(k, 0)$, with $k \in \mathbb{N}$, the intertwining operators among the modules have the associativity property, the category has a natural structure of vertex tensor category, and a number of other important results hold. For the tensor product theory for modules for a vertex operator algebra, see [HL1]–[HL6] and [H1]–[H2].

We need the following notion introduced in [H1].

Definition 3.1. Let V be a vertex operator algebra. We say that products of intertwining operators for V satisfy the *convergence and extension property* if for any intertwining operators \mathfrak{y}_1 and \mathfrak{y}_2 , of types $\binom{W_0}{W_1 W_4}$ and $\binom{W_4}{W_2 W_3}$, respectively, there exists an integer N (depending only on \mathfrak{y}_1 and \mathfrak{y}_2) such that the following condition holds: For any $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$, and $w'_{(0)} \in W'_0$, with $w_{(1)}$ and $w_{(2)}$ homogeneous, there exist $j \in \mathbb{N}$, $r_i, s_i \in \mathbb{R}$, for $i = 1, \dots, j$, and analytic functions $f_i(z)$ on $|z| < 1$, for $i = 1, \dots, j$, satisfying

$$\text{wt } w_{(1)} + \text{wt } w_{(2)} + s_i > N, \quad \text{for } i = 1, \dots, j, \quad (3.1)$$

such that

$$\left\langle w'_{(0)}, \mathcal{Y}_1(w_{(1)}, x_1) \mathcal{Y}_2(w_{(2)}, x_2) w_{(3)} \right\rangle \Big|_{x_i^r = e^{r \log z_i}, i=1,2, r \in \mathbb{C}} \quad (3.2)$$

is absolutely convergent when $|z_1| > |z_2| > 0$ and can be analytically extended to the multivalued analytic function

$$\sum_{i=1}^j z_2^{r_i} (z_1 - z_2)^{s_i} f_i \left(\frac{z_1 - z_2}{z_2} \right) \quad (3.3)$$

when $|z_2| > |z_1 - z_2| > 0$; here “log” denotes the standard branch of the log function.

Note that if the associativity of intertwining operators holds for V (see [H1]–[H2], and Theorem 3.8 below), then products of intertwining operators satisfy the convergence and extension property, so that this condition is necessary for the associativity of intertwining operators; to see that the expression in (3.8) has the form (3.3), use the $L(0)$ -conjugation formula (5.4.22) in [FHL].

We shall also use the concepts of generalized module as defined in [HL3] and of weak module as defined in [DLM]. Let V be a vertex operator algebra. A *generalized V -module* is a \mathbb{C} -graded vector space equipped with a vertex operator map satisfying all the axioms for a V -module except the two grading-restriction axioms. If there exists $N \in \mathbb{Z}$ such that the homogeneous subspace of weight n of a generalized V -module is 0, when the real part of n is less than N , the generalized V -module is said to be *lower truncated*.

A *weak V -module* is a vector space equipped with a vertex operator map satisfying all the axioms for a V -module except those axioms involving a grading. Dong, Li, and Mason [DLM] have proved a stronger result than Theorem 1.3—that for any $k \in \mathbb{N}$, a weak $L(k, 0)$ -module is completely reducible and all the irreducible weak $L(k, 0)$ -modules are in fact $L(k, 0)$ -modules, as listed in Theorem 1.3. In particular, a finitely generated lower-truncated generalized $L(k, 0)$ -module is a module (see also [FZ] and [L2], which use instead the notion of module in [FZ]).

We also have that the tensor product vertex operator algebra $L(k_1, 0) \otimes \cdots \otimes L(k_m, 0)$, for any $k_1, \dots, k_m \in \mathbb{N}$, is rational, by the stronger result just stated and by arguments in [FHL, Section 4.7] and [DMZ, Section 2]; see also [L1]. Actually, for the complete reducibility of a module for the tensor product vertex operator algebra, one can alternatively argue, as in [FZ], [L2], and [DMZ], using the notion of module in [FZ] rather than the notion of weak module. (An argument essentially the same as this complete reducibility argument appears in the proof of condition (1) of Theorem 3.2.)

The following result establishes the remaining conditions for the applicability of the tensor product theory (see [HL1]–[HL6] and [H1]–[H2]) to the vertex operator algebra $L(k_1, 0) \otimes \cdots \otimes L(k_m, 0)$.

THEOREM 3.2. *For any $m \in \mathbb{Z}_+$ and $k_i \in \mathbb{N}$, with $i = 1, \dots, m$, we have the following.*

(1) *Every finitely generated lower-truncated generalized module for the vertex operator algebra $L(k_1, 0) \otimes \dots \otimes L(k_m, 0)$ is a module.*

(2) *Products of intertwining operators for $L(k_1, 0) \otimes \dots \otimes L(k_m, 0)$ have the convergence and extension property.*

(3) *For any modules W_j , with $j = 1, \dots, 2n + 1$, for $L(k_1, 0) \otimes \dots \otimes L(k_m, 0)$; any intertwining operators \mathcal{Y}_i , with $i = 1, \dots, n$, of types $\binom{W_{2i-1}}{W_{2i}W_{2i+1}}$, respectively; and any $w'_{(1)} \in W'_1$, $w_{(2i)} \in W_{2i}$, with $i = 1, \dots, n$, and $w_{(2n+1)} \in W_{2n+1}$,*

$$\left\langle w'_{(1)}, \mathcal{Y}_1(w_{(2)}, x_1) \cdots \mathcal{Y}_n(w_{(2n)}, x_n) w_{(2n+1)} \right\rangle \Big|_{x_i^r = e^{r \log z_i}, 1 \leq i \leq n, r \in \mathbb{C}} \tag{3.4}$$

is absolutely convergent for any $z_1, \dots, z_n \in \mathbb{C}$ satisfying $|z_1| > \dots > |z_n| > 0$; here the choice of $\log z_i$ is arbitrary for each i .

Proof. We first prove the theorem in the case $m = 1$. We have already verified the first condition in this case.

To prove that products of intertwining operators for $L(k, 0)$ have the convergence and extension property, we can assume that the W_l , for $l = 0, \dots, 4$, are irreducible since $L(k, 0)$ is rational. Let $\lambda_l \in \mathfrak{h}^*$, for $l = 0, \dots, 3$, be dominant integral weights satisfying $(\lambda_l, \theta) \leq k$; let W be a V -module; and let \mathcal{Y}_1 and \mathcal{Y}_2 be intertwining operators of types $\binom{L(k, \lambda_0)}{L(k, \lambda_1)W}$ and $\binom{W}{L(k, \lambda_2)L(k, \lambda_3)}$, respectively. For $w'_{(0)} \in L(\lambda_0)^* \subset L(k, \lambda_0)'$ and $w_{(i)} \in L(\lambda_i) \subset L(k, \lambda_i)$, for $i = 1, 2, 3$, we have already shown in the preceding section that (3.2) satisfies the KZ equations (2.15) and (2.16). Thus it is absolutely convergent when $|z_1| > |z_2| > 0$. The KZ equations have regular singular points. Using the theory of such equations (cf. [K]), expression (3.2), as a solution of the KZ equations in the region given by this inequality, can be analytically extended to a solution in the region given by the inequality $|z_2| > |z_1 - z_2| > 0$, and the extension can be expanded as a suitably truncated series in rational powers of z_2 , $z_1 - z_2$, $\log z_2$, and $\log(z_1 - z_2)$. Since, by the definition of intertwining operator, the original solution does not contain $\log z_1$ or $\log z_2$, the extension cannot contain $\log z_2$ or $\log(z_1 - z_2)$, and so the extension must be of the form (3.3). We take N to be an integer such that (3.1) holds for the $w_{(i)}$ and $w'_{(0)}$ above and for the s_i , with $i = 1, \dots, j$, appearing in the extension (3.3). Then N depends only on \mathcal{Y}_1 and \mathcal{Y}_2 . (This argument is the same as that in [TK] and in the proof of Theorem 3.5 in [H2].)

For general elements $w'_{(0)} \in L(k, \lambda_0)'$ and $w_{(l)} \in L(k, \lambda_l)$, with $l = 1, 2, 3$, we note that since the universal enveloping algebra $U(\hat{\mathfrak{g}}_-)$ is in fact generated by the $g(-1)$ for $g \in \mathfrak{g}$, the general elements are linear combinations of elements of the form $g_1(-1) \cdots g_m(-1)w$, where w is a lowest conformal-weight vector and $g_i \in \mathfrak{g}$. Using (2.3), (2.10), and (2.11), we easily see that

$$\left\langle w'_{(0)}, \mathcal{Y}_1(w_{(1)}, x_1) \mathcal{Y}_2(w_{(2)}, x_2) w_{(3)} \right\rangle \tag{3.5}$$

can be written as a linear combination of expressions of the same form but with $w'_{(0)} \in L(\lambda_0)^*$, $w_{(l)} \in L(\lambda_l)$, for $l = 1, 2, 3$, and with Laurent polynomials in x_1, x_2 , and $x_1 - x_2$ (suitably expanded) as coefficients. In particular, the product $\mathfrak{Y}_1(\cdot, x_1)\mathfrak{Y}_2(\cdot, x_2)$ of intertwining operators is determined by the expressions (3.5) with the four vectors in the lowest conformal-weight spaces. Moreover, it is easy to see that for general $w'_{(0)}$, $w_{(l)}$, with $l = 1, 2, 3$, expression (3.2) is absolutely convergent when $|z_1| > |z_2| > 0$ and can be extended to a multivalued analytic function of the form (3.3) when $|z_2| > |z_1 - z_2| > 0$; note that for $n > 0$, x_1^{-n} is to be written as $x_2^{-n}(1 + (x_1 - x_2)/x_2)^{-n}$. In addition, each application of $g(-1)$ to $w_{(1)}$ or $w_{(2)}$ introduces at most one expression $(x_1 - x_2)^{-1}$, and each application of $g(-1)$ to $w'_{(0)}$ or $w_{(3)}$ introduces no such poles. We make this explicit (proving the inequality (3.1)) by using induction on the sum of the weights of $w'_{(0)}$ and the $w_{(l)}$, with $l = 1, 2, 3$.

Assume that the convergence and extension property holds when

$$\text{wt } w'_{(0)} + \text{wt } w_{(1)} + \text{wt } w_{(2)} + \text{wt } w_{(3)} < q.$$

Any homogeneous element of $L(k, \lambda)$ is either a lowest weight vector or a linear combination of elements of the form $g(-1)\tilde{w}$, where \tilde{w} has lower weight. Thus, when

$$\text{wt } w'_{(0)} + \text{wt } w_{(1)} + \text{wt } w_{(2)} + \text{wt } w_{(3)} = q,$$

at least one of the elements $w'_{(0)}$, $w_{(1)}$, $w_{(2)}$, and $w_{(3)}$ is a linear combination of elements of the form $g(-1)\tilde{w}$. We prove only the case $w_{(1)} = g(-1)\tilde{w}_{(1)}$ for some $\tilde{w}_{(1)} \in L(k, \lambda_1)$, the other cases being similar.

By (2.3),

$$\mathfrak{Y}_1(g(-1)\tilde{w}_{(1)}, x_1) = \circ g(x_1)\mathfrak{Y}_1(\tilde{w}_{(1)}, x_1)\circ,$$

and so by (2.10),

$$\begin{aligned} & \left\langle w'_{(0)}, \mathfrak{Y}_1(w_{(1)}, x_1)\mathfrak{Y}_2(w_{(2)}, x_2)w_{(3)} \right\rangle \\ &= - \sum_{p \in \mathbb{N}} x_1^p \left\langle g(p+1)w'_{(0)}, \mathfrak{Y}_1(\tilde{w}_{(1)}, x_1)\mathfrak{Y}_2(w_{(2)}, x_2)w_{(3)} \right\rangle \\ & \quad + \sum_{p \in \mathbb{N}} x_1^{-p-1} \left\langle w'_{(0)}, \mathfrak{Y}_1(\tilde{w}_{(1)}, x_1)\mathfrak{Y}_2(w_{(2)}, x_2)g(p)w_{(3)} \right\rangle \quad (3.6) \\ & \quad + (x_1 - x_2)^{-1} \left\langle w'_{(0)}, \mathfrak{Y}_1(\tilde{w}_{(1)}, x_1)\mathfrak{Y}_2(gw_{(2)}, x_2)w_{(3)} \right\rangle. \end{aligned}$$

Note that the sums are finite. Since the sum of the weights of the module elements in each term of the right-hand side of (3.6) is less than q , by the induction assumption we obtain the desired conclusion for elements the sum of whose weights is q . This proves that products of intertwining operators for $L(k, 0)$ have the convergence and extension property.

Using the KZ equations (2.14), the same method as for products of two intertwining operators shows that (3.4) is absolutely convergent when $|z_1| > \cdots > |z_n| > 0$.

We now prove the general case ($m \geq 1$). Let $m > 1$ and $k_i \in \mathbb{N}$, for $i = 1, \dots, m$. Let W be a lower-truncated generalized module for $L(k_1, 0) \otimes \cdots \otimes L(k_m, 0)$. We shall actually prove that W is a direct sum of irreducible modules, using arguments of [FHL] and [DMZ]. Any element $w \in W$ generates a weak $L(k_i, 0)$ -module, for $i = 1, \dots, m$. Recall that for any $i = 1, \dots, m$, any weak $L(k_i, 0)$ -module is completely reducible, and the irreducible weak $L(k_i, 0)$ -modules are in fact the $L(k_i, 0)$ -modules listed in Theorem 1.3. Thus the weak $L(k_i, 0)$ -module generated by w is a finite direct sum of irreducible $L(k_i, 0)$ -modules. So the tensor product of these $L(k_i, 0)$ -modules, for $i = 1, \dots, m$, as an $L(k_1, 0) \otimes \cdots \otimes L(k_m, 0)$ -module, is a finite direct sum of tensor products of irreducible $L(k_i, 0)$ -modules, for $i = 1, \dots, m$. By [FHL, Proposition 4.7.2], these tensor products of irreducible modules are irreducible modules for $L(k_1, 0) \otimes \cdots \otimes L(k_m, 0)$, and so we have a finite direct sum of irreducible $L(k_1, 0) \otimes \cdots \otimes L(k_m, 0)$ -modules, which canonically maps onto the generalized $L(k_1, 0) \otimes \cdots \otimes L(k_m, 0)$ -module generated by w . This in turn is a finite direct sum of modules, and so W is completely reducible. If W is finitely generated, it must be a module for $L(k_1, 0) \otimes \cdots \otimes L(k_m, 0)$. (This argument also works using the notion of module in [FZ] in place of the notion of weak module; cf. [DMZ, Proposition 2.7].)

Recall that for a vertex operator algebra V and V -modules W_1, W_2 , and W_3 , the fusion rule $\mathcal{N}_{W_1 W_2}^{W_3}$ is the dimension of space $\mathcal{V}_{W_1 W_2}^{W_3}$ of all intertwining operators of type $\binom{W_3}{W_1 W_2}$ (see [FHL]). By [DMZ, Proposition 2.10] (see also [L1]), we have the following: Let $m > 0$, $k_i \in \mathbb{N}$, $V = L(k_1, 0) \otimes \cdots \otimes L(k_m, 0)$, and $W_t = L(k_1, \lambda_1^{(t)}) \otimes \cdots \otimes L(k_m, \lambda_m^{(t)})$, for $t = 1, 2, 3$, irreducible V -modules (recall [FHL, Theorem 4.7.4]), and let \mathcal{Y} be an intertwining operator of type $\binom{W_3}{W_1 W_2}$. For convenience, we shall write the fusion rules $\mathcal{N}_{L(k_i, \lambda_i^{(1)}) L(k_i, \lambda_i^{(2)})}^{L(k_i, \lambda_i^{(3)})}$, for $i = 1, \dots, m$, as $\mathcal{N}_{12}^{3;i}$. Then there exist intertwining operators $\mathcal{Y}_i^{(l_i)}$ of type $\binom{L(k_i, \lambda_i^{(3)})}{L(k_i, \lambda_i^{(1)}) L(k_i, \lambda_i^{(2)})}$, $l_i = 1, \dots, \mathcal{N}_{12}^{3;i}$, for $i = 1, \dots, m$, such that

$$\mathcal{Y} = \sum_{l_1=1}^{\mathcal{N}_{12}^{3;1}} \cdots \sum_{l_m=1}^{\mathcal{N}_{12}^{3;m}} \mathcal{Y}_1^{(l_1)} \otimes \cdots \otimes \mathcal{Y}_m^{(l_m)}, \quad (3.7)$$

where both the left- and right-hand sides are understood as linear maps from $W_1 \otimes W_2$ to $W_3\{x\}$.

This result reduces the convergence and extension property for products of intertwining operators for the vertex operator algebra $L(k_1, 0) \otimes \cdots \otimes L(k_m, 0)$ to the corresponding properties that we have proved above (as in the proof of Theorem 3.5 in [H2]). Similarly, the third conclusion of the theorem follows immediately from

(3.7) and the case $m = 1$ proved above. □

Remark 3.3. The proof of the convergence and extension property for products of two intertwining operators and the proof of the convergence of products of an arbitrary number of intertwining operators are basically the same as in [TK]. But Tsuchiya and Kanie analytically extended these convergent products of an arbitrary number of intertwining operators to the whole configuration space and had to show that there are no logarithms of the variables occurring in the extensions, in order to construct the braid group representations. In our (more general) theory, we need only prove the extension property of products of two intertwining operators, and also, this extension property requires only that products can be extended to the region $|z_2| > |z_1 - z_2| > 0$. The extension property for products of an arbitrary number of intertwining operators to the whole configuration space then follows from the convergence of the same number of intertwining operators, the associativity of intertwining operators (proved using the extension property for products of two intertwining operators) and skew-symmetry, and other properties of intertwining operators.

We now define the following class of vertex operator algebras.

Definition 3.4. Let $m \in \mathbb{Z}_+$ and $k_i \in \mathbb{N}$. A vertex operator algebra V is said to be in the class $\mathcal{L}_{k_1, \dots, k_m}$ if V has a vertex operator subalgebra (with the same Virasoro element as V) isomorphic to $L(k_1, 0) \otimes \cdots \otimes L(k_m, 0)$.

For vertex operator algebras in this class, by [H2, Theorem 3.2] and Theorem 3.2 above, the condition for the applicability of the tensor product theory remaining to be verified is the following.

PROPOSITION 3.5. *Let V be a vertex operator algebra in the class $\mathcal{L}_{k_1, \dots, k_m}$. Then every finitely generated lower-truncated generalized V -module is a V -module.*

Proof. The space V is a module for the (rational) vertex operator algebra $L(k_1, 0) \otimes \cdots \otimes L(k_m, 0)$ and so is a finite direct sum of tensor products of irreducible $L(k_i, 0)$ -modules by [FHL, Theorem 4.7.4]. Since irreducible $L(k_i, 0)$ -modules are spanned by elements of the form $g_1(-1) \cdots g_m(-1)v$, where v is a lowest conformal-weight vector and $g_i \in \mathfrak{g}$, V is spanned by tensor products of elements of this form, where the needed lowest conformal-weight vectors range through a finite-dimensional subspace of V . Let W be a lower-truncated generalized V -module generated by a single element w . Then W is also a lower-truncated generalized $L(k_1, 0) \otimes \cdots \otimes L(k_m, 0)$ -module, and the $L(k_1, 0) \otimes \cdots \otimes L(k_m, 0)$ -submodule generated by w is a module by Theorem 3.2. By a lemma of Dong and Mason [DM] and Li [L1], W is spanned by elements of the form $u_n w$, where $u \in V$ and $n \in \mathbb{Z}$. Thus from (2.3) for the vertex operator map associated with W , we easily see that W is a module. (Cf. the proof of Proposition 3.7 in [H2].) □

Let V be a vertex operator algebra in the class $\mathcal{L}_{k_1, \dots, k_m}$. For the conformal equivalence class $P(z)$ of spheres with negatively oriented puncture ∞ and positively

oriented punctures z and 0 and with the trivial local coordinates vanishing at these punctures, recall the bifunctor $\boxtimes_{P(z)}$ and the tensor product bifunctor $\boxtimes_{P(z)}$ constructed in [HL5]. By Theorem 3.2, Proposition 3.5, and [H2, Theorems 3.1, 3.2, and Corollary 3.3], which in turn are proved using results in [HL1]–[HL6] and [H1], we obtain the following results.

PROPOSITION 3.6. *For any V -modules W_1 and W_2 , $W_1 \boxtimes_{P(z)} W_2$ is a module, and the $P(z)$ -product $(W_1 \boxtimes_{P(z)} W_2, Y_{P(z)}; \boxtimes_{P(z)})$ is a $P(z)$ -tensor product of W_1 and W_2 .*

THEOREM 3.7. *For any V -modules W_1 , W_2 , and W_3 and any complex numbers z_1 and z_2 satisfying $|z_1| > |z_2| > |z_1 - z_2| > 0$, there exists a unique isomorphism $\mathcal{A}_{P(z_1), P(z_2)}^{P(z_1-z_2), P(z_2)}$ from $W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)$ to $(W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3$ such that for any $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, and $w_{(3)} \in W_3$,*

$$\overline{\mathcal{A}_{P(z_1), P(z_2)}^{P(z_1-z_2), P(z_2)}} \left(w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} w_{(3)}) \right) = (w_{(1)} \boxtimes_{P(z_1-z_2)} w_{(2)}) \boxtimes_{P(z_2)} w_{(3)},$$

where

$$\overline{\mathcal{A}_{P(z_1), P(z_2)}^{P(z_1-z_2), P(z_2)}} : \overline{W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)} \rightarrow \overline{(W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3}$$

is the canonical extension of $\mathcal{A}_{P(z_1), P(z_2)}^{P(z_1-z_2), P(z_2)}$.

THEOREM 3.8 (Associativity for intertwining operators). (1) *For any V -modules W_0 , W_1 , W_2 , W_3 , and W_4 , any intertwining operators \mathfrak{Y}_1 and \mathfrak{Y}_2 of types $\binom{W_0}{W_1 W_4}$ and $\binom{W_4}{W_2 W_3}$, respectively, and any choice of $\log z_1$ and $\log z_2$,*

$$\left\langle w'_{(0)}, \mathfrak{Y}_1(w_{(1)}, x_1) \mathfrak{Y}_2(w_{(2)}, x_2) w_{(3)} \right\rangle \Big|_{x_1^n = e^{n \log z_1}, x_2^n = e^{n \log z_2}, n \in \mathbb{C}}$$

is absolutely convergent when $|z_1| > |z_2| > 0$, for $w'_{(0)} \in W'_0$, $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, and $w_{(3)} \in W_3$. For any modules W_0 , W_1 , W_2 , W_3 , and W_5 , any intertwining operators \mathfrak{Y}_3 and \mathfrak{Y}_4 of types $\binom{W_5}{W_1 W_2}$ and $\binom{W_0}{W_5 W_3}$, respectively, and any choice of $\log z_2$ and $\log(z_1 - z_2)$,

$$\left\langle w'_{(0)}, \mathfrak{Y}_4 \left(\mathfrak{Y}_3(w_{(1)}, x_0) w_{(2)}, x_2 \right) w_{(3)} \right\rangle \Big|_{x_0^n = e^{n \log(z_1 - z_2)}, x_2^n = e^{n \log z_2}, n \in \mathbb{C}}$$

is absolutely convergent when $|z_2| > |z_1 - z_2| > 0$, for $w'_{(0)} \in W'_0$, $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, and $w_{(3)} \in W_3$.

(2) *For any V -modules W_0 , W_1 , W_2 , W_3 , and W_4 and any intertwining operators \mathfrak{Y}_1 and \mathfrak{Y}_2 of types $\binom{W_0}{W_1 W_4}$ and $\binom{W_4}{W_2 W_3}$, respectively, there exist a module W_5 and intertwining operators \mathfrak{Y}_3 and \mathfrak{Y}_4 of types $\binom{W_5}{W_1 W_2}$ and $\binom{W_0}{W_5 W_3}$, respectively, such that*

for any $z_1, z_2 \in \mathbb{C}$ satisfying $|z_1| > |z_2| > |z_1 - z_2| > 0$ and any fixed choices of $\log z_1$, $\log z_2$, and $\log(z_1 - z_2)$,

$$\begin{aligned} & \left\langle w'_{(0)}, \mathfrak{Y}_1(w_{(1)}, x_1) \mathfrak{Y}_2(w_{(2)}, x_2) w_{(3)} \right\rangle \Big|_{x_1^n = e^{n \log z_1}, x_2^n = e^{n \log z_2}, n \in \mathbb{C}} \\ &= \left\langle w'_{(0)}, \mathfrak{Y}_4 \left(\mathfrak{Y}_3(w_{(1)}, x_0) w_{(2)}, x_2 \right) w_{(3)} \right\rangle \Big|_{x_0^n = e^{n \log(z_1 - z_2)}, x_2^n = e^{n \log z_2}, n \in \mathbb{C}}, \end{aligned} \quad (3.8)$$

for any $w'_{(0)} \in W'_0$, $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, and $w_{(3)} \in W_3$.

(3) For any modules W_0, W_1, W_2, W_3 , and W_5 and any intertwining operators \mathfrak{Y}_3 and \mathfrak{Y}_4 of types $\binom{W_5}{W_1 W_2}$ and $\binom{W_0}{W_5 W_3}$, respectively, there exist a module W_4 and intertwining operators \mathfrak{Y}_1 and \mathfrak{Y}_2 of types $\binom{W_0}{W_1 W_4}$ and $\binom{W_4}{W_2 W_3}$, respectively, such that for any $z_1, z_2 \in \mathbb{C}$ satisfying $|z_1| > |z_2| > |z_1 - z_2| > 0$ and any fixed choices of $\log z_1$, $\log z_2$, and $\log(z_1 - z_2)$, (3.8) holds for any $w'_{(0)} \in W'_0$, $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, and $w_{(3)} \in W_3$.

THEOREM 3.9 (Commutativity for intertwining operators). *For any V -modules W_0, W_1, W_2, W_3 , and W_4 and any intertwining operators \mathfrak{Y}_1 and \mathfrak{Y}_2 of types $\binom{W_0}{W_1 W_4}$ and $\binom{W_4}{W_2 W_3}$, respectively, there exist a module W_5 and intertwining operators \mathfrak{Y}_3 and \mathfrak{Y}_4 of types $\binom{W_0}{W_2 W_5}$ and $\binom{W_5}{W_1 W_3}$, respectively, such that for any $w'_{(0)} \in W'_0$, $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, and $w_{(3)} \in W_3$, the multivalued analytic function*

$$\left\langle w'_{(0)}, \mathfrak{Y}_1(w_{(1)}, x_1) \mathfrak{Y}_2(w_{(2)}, x_2) w_{(3)} \right\rangle \Big|_{x_1 = z_1, x_2 = z_2}$$

of z_1 and z_2 in the region $|z_1| > |z_2| > 0$ and the multivalued analytic function

$$\left\langle w'_{(0)}, \mathfrak{Y}_3(w_{(2)}, x_2) \mathfrak{Y}_4(w_{(1)}, x_1) w_{(3)} \right\rangle \Big|_{x_1 = z_1, x_2 = z_2}$$

of z_1 and z_2 in the region $|z_2| > |z_1| > 0$ are analytic extensions of each other.

In [H3], the notion of intertwining operator algebra was introduced (see also [H5] and [H6]). By [H3, Theorem 3.5], we obtain the following.

THEOREM 3.10. *Assume that V is rational. Let $\mathcal{A} = \{a_i\}_{i=1}^m$ be the set of all equivalence classes of irreducible V -modules, and let W^{a_1}, \dots, W^{a_m} be representatives of a_1, \dots, a_m , respectively. Let $W = \coprod_{i=1}^m W^{a_i}$, and let $\mathfrak{V}_{a_i a_j}^{a_k}$, for $a_i, a_j, a_k \in \mathcal{A}$, be the space of intertwining operators of type $\binom{W^{a_k}}{W^{a_i} W^{a_j}}$. Then $(W, \mathcal{A}, \{\mathfrak{V}_{a_i a_j}^{a_k}\}, \mathbf{1}, \omega)$ (where $\mathbf{1}$ and ω are the vacuum and Virasoro element of V) is an intertwining operator algebra.*

Recall the sphere partial operad $K = \{K(j)\}_{j \in \mathbb{N}}$, the vertex partial operads $\tilde{K}^c = \{\tilde{K}^c(j)\}_{j \in \mathbb{N}}$ of central charge $c \in \mathbb{C}$ constructed in [H4], and the definition of vertex tensor category in [HL2] and [HL6]. For any $c \in \mathbb{C}$ and $j \in \mathbb{N}$, $\tilde{K}^c(j)$ is a trivial

holomorphic line bundle over $K(j)$, and we have a canonical holomorphic section ψ_j . Given a vertex tensor category, we have, among other things, a tensor product bifunctor $\boxtimes_{\tilde{Q}}$ for each $\tilde{Q} \in \tilde{K}^c(2)$. In particular, $\psi_2(P(z)) \in \tilde{K}^c(2)$, and thus there is a tensor product bifunctor $\boxtimes_{\psi_2(P(z))}$.

THEOREM 3.11. *Let c be the central charge of V . Then the category of V -modules has a natural structure of vertex tensor category of central charge c such that for each $z \in \mathbb{C}^\times$, the tensor product bifunctor $\boxtimes_{\psi_2(P(z))}$ associated with $\psi_2(P(z)) \in \tilde{K}^c(2)$ is equal to $\boxtimes_{P(z)}$ constructed in [HL5].*

Combining Theorem 3.11 with [HL2, Theorem 4.4] (see [HL6] for the proof), we obtain the following corollary.

COROLLARY 3.12. *Let V be a vertex operator algebra in the class $\mathcal{L}_{k_1, \dots, k_m}$. Then the category of V -modules has a natural structure of braided tensor category such that the tensor product bifunctor is $\boxtimes_{P(1)}$. In particular, the category of $L(k_1, 0) \otimes \cdots \otimes L(k_m, 0)$ -modules has a natural structure of braided tensor category.*

The special case $V = L(k, 0)$ states (recall the terminology defined at the beginning of the introduction) the following theorem.

THEOREM 3.13. *For $k \in \mathbb{N}$, the category generated by the standard $\hat{\mathfrak{g}}$ -modules of level k has a natural structure of braided tensor category such that the tensor product bifunctor is $\boxtimes_{P(1)}$.*

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